On Normalization of Proofs in Set Theory

by

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A Preprint from University of Stockholm
Department of Philosophy

1983

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Preprint N:o 1, 1983

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ISBN 91-7146-321-6
Holms Gårds tryckeri, Edsbruk, 1983

ACKNOWLEDGEMENTS

The subject was proposed by my thesis adviser, Dag Prawitz.

Besides Prawitz, the paper has been discussed in detail with Torkel Franzén. Their criticism and suggestions for improvements have helped me considerably to find better solutions to problems and better presentations of various ideas and proofs. This is particularly true for section 4 of chapter II, which has been completely rewritten a number of times as a result of their criticism.

Besides the work of Prawitz this paper is to a considerable extent inspired by Per Martin-Löf's beautiful work on normalization in systems of natural deduction.

Anna Britta Jansson has typed the manuscript and has been a great help in preparing it for the printers.

To all of those I extend my heartfelt THANKS!

Stockholm, May 1983

Lars Hallnäs

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INTRODUCTION

When we make a formal analysis of something informally given we naturally ask ourselves whether the proposed analysis captures what it was meant to capture or not. Perhaps it turns out to be rather difficult to find more precise criteria for the correctness of an analysis of the kind we have put forward. Searching for such criteria we can look for certain tests that must be passed if the analysis is not to be in a certain sense inadequate. Rather than find criteria concerning all possible aspects we may test the analysis from different points of view.

When it comes to making a formal analysis of proofs in a given informal practice there are several such basic tests. Soundness and completeness are two such tests. If, following the tradition that goes back to Gentzen, the analysis is carried out in terms of a system of natural deduction, the normalization procedures introduced by Prawitz provide another such test of the adequacy of the analysis. These procedures give a formal test of how well the system represents the structure of the informally given proofs: if we are dealing with a sound proof and our system has captured the structure of this proof well enough, then certain nontrivial operations on the deduction representing the proof ought to be wellfounded and as a consequence this deduction should be normalizable.

So normalization of deductions can be viewed as a formal first step in investigations of the structure-representation of proofs in natural deduction.

In the present paper I shall formulate a canonical natural deduction system for naive set theory and investigate the nature of the proof theoretical analysis of set theory that this system gives. The investigation will take the form of a study of normalization properties of fragments of naive set theory, and therefore it will be focused on the problem of structural representation of set theoretical proofs in naive set theory.

In view of the inconsistency of this theory it might seem a little strange to take this system as a basis for analyzing set theoretical proofs. What one should remember here is that in any reasonable formal analysis of set theory, the rules of our formal system of naive set theory should be derivable. The simplicity of the rules, together with the fact that it is hard to find rules more elementary, makes it natural to ask how far this analysis goes.

This paper is devided into three chapters. In the first chapter, besides the formulation of the basic system itself, I will give rules of contraction as an explication of certain syntactical properties of the rules of inference. Counter examples to normalizability in the basic system are also given. One of these counter examples, stemming from Marcel Crabbe, shows that full normalization is not even possible in fragments based on Zermelo's subset axiom.

In the second chapter I will define a class of fragments, the well-founded fragments, of naive set theory. I will show that there is a canonical formulation of ZF in terms of a system of natural deduction ZF_N and that ZF_N is a wellfounded fragment of naive set theory provided there is a wellfounded model of ZF. A modified concept of normalizability, $\mathfrak{C}$-normalizability, will be discussed. Although this concept is weaker than normalizability, it still preserves the basic property that closed $\mathfrak{C}$-normal deductions are on introduction form. This concept is introduced in order to avoid the problems raised by the counter examples to normalizability given in the first chapter. A semantical motivation for the rules of contraction associated with $\mathfrak{C}$-normalization will be presented for wellfounded fragments of naive set theory.

In the third chapter I will prove that deductions in wellfounded fragments of naive set theory are all $\mathfrak{C}$-normalizable. The method I shall use to prove this result is an extension of methods used by Girard and Martin-Löf in their proofs of normalizability in simple type theory, and by Prawitz in his proof of normalizability in second order classical logic.

Throughout this paper the main treatment will be focused on a system of natural deduction based on classical logic. It will also be indicated how to obtain a treatment in case our system should be based on intuitionistic logic. Thus the results will be obtained both for systems based on classical logic and for systems based on intuitionistic logic.

The reason for choosing classical logic as a basis for the main treatment is simply that the main area of set theoretical practice is based on the use of the law of excluded middle without restrictions in proving mathematical theorems.

I NAIVE SET THEORY

1 The formal system

1.1 Terms and formulas

Variables x, y, z... are terms.
$\perp$ is a formula.
If r, s are terms, then $r \in t$, $r = t$ are formulas.
If A, B are formulas, then A & B,
$A \to B$ are formulas.
If A(x) is a formula, then $\{x \mid A(x)\}$ is a term and $\forall x\, A(x)$ a formula.

Formulas that differ only in bound variables are identified.
Formulas of the form $r \in x$ and $\perp$ are called atomic formulas.
Other formulas are referred to as composite formulas.
$A(t)$ indicates the formula obtained by substituting t for x in $A(x)$.

$$\neg A \underset{def.}{=} A \to \perp$$

$$A \vee B \underset{def.}{=} \neg A \to B$$

$$\exists x A(x) \underset{def.}{=} \neg \forall x \neg A(x)$$

1.2 Deductions and rules of inference

1.2.1 The formal system for naive set theory considered here is obtained by adding some new rules of inference to an ordinary system of natural deduction for first order logic. So deductions in naive set theory are built up from assumptions using the following rules of inference:

→introduction (→I)

$$\dfrac{\begin{array}{c}\not{A}\\ \vdots\\ B\end{array}}{A \to B}$$

$\not{A}$ indicates that some occurrences of A as an assumption in the deduction

$$\begin{array}{c}A\\ \vdots\\ B\end{array}$$

may have been cancelled at the inference of A B. An assumption that has not been cancelled is called open.

→elimination (→E)

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ A & A \to B\end{array}}{B}$$

& introduction (&I)

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ A & B\end{array}}{A \& B}$$

& elimination (&E)

$$\dfrac{\begin{array}{c}\vdots\\ A \& B\end{array}}{A} \qquad \dfrac{\begin{array}{c}\vdots\\ A \& B\end{array}}{B}$$

$\forall$ introduction $(\forall I)$

$$\frac{\begin{array}{c}\vdots\\ A(x)\end{array}}{\forall x A(x)}$$

where x must not occur in any open assumption in the deduction

$$\begin{array}{c}\vdots\\ A(x)\end{array}$$

$\forall$ elimination $(\forall E)$

$$\frac{\begin{array}{c}\vdots\\ \forall x\, A(x)\end{array}}{A(t)}$$

$\in$ introduction $(\in I)$

$$\frac{\begin{array}{c}\vdots\\ A(r)\end{array}}{r \in \{x \mid A(x)\}}$$

$\in$ elimination $(\in E)$

$$\frac{\begin{array}{c}\vdots\\ r \in \{x \mid A(x)\}\end{array}}{A(r)}$$

$=$ introduction $(=I)$

$$\frac{\begin{array}{cc}\not{x \in S} & \not{x \in r}\\ \vdots & \vdots\\ x \in r & x \in S\end{array}}{S = r}$$

where x must not occur in any open assumptions other than $x \in S$, $x \in r$ in the deductions

$$\begin{array}{cc}x \in S & x \in r\\ \vdots & \vdots\\ x \in r & x \in S\end{array}$$

$=$ elimination $(=E)$

$$\begin{array}{c} \vdots \quad \vdots \\ (A(x)) \dfrac{A(s) \quad s=r}{A(r)} \end{array} \qquad \begin{array}{c} \vdots \quad \vdots \\ (A(x)) \dfrac{A(r) \quad s=r}{A(s)} \end{array}$$

$\perp$ elimination $(\perp E)$

$$\begin{array}{c} \not\neg A \\ \vdots \\ \perp \\ \hline A \end{array}$$

In the $\rightarrow E$ inferences $A \rightarrow B$ will be called the <u>major</u> premise and A the <u>minor</u> premise. In the $=E$ inferences $s=r$ is the major premise and $A(s)$, $A(r)$ the minor premise. For other elimination inferences the one premise will be called the major premise. And in elimination inferences the deduction of the major premise will be called the <u>major</u> deduction of the inference and the deduction of the minor premise will be called the <u>minor</u> deduction of the inference.

If a deduction $\mathcal{D}$ ends with A and has $\Gamma = \{A_1 \ldots A_n\}$ as open assumptions this deduction is a deduction of A from Γ. This will be written $\mathcal{D}(\Gamma; A)$ and occasionally also as

$$\begin{array}{c} A_1 \ldots A_n \\ \vdots \\ A \end{array}$$

If Γ is empty $\mathcal{D}$ is a <u>closed</u> deduction and we write simply $\mathcal{D}(A)$. If Γ is non-empty $\mathcal{D}$ is <u>open</u>.

The notation $\mathcal{D}(x)$ indicates that x may occur free in the endformula of $\mathcal{D}$ and if so x is a free variable in $\mathcal{D}$. $\mathcal{D}(t/x)$ is obtained from $\mathcal{D}$ by substituting t for the free variable x in $\mathcal{D}$ at occurrences where x is free for substitution. If $\mathcal{D}$ is a deduction that ends with an I-inference I will say that $\mathcal{D}$ is on I-form and similarly if $\mathcal{D}$ ends with an E-inference that $\mathcal{D}$ is on E-form. A composite application of a rule of inference Φ is a deduction whose endformula is composite and whose last inference is Φ, except when Φ is $=E$; in this case a composite application is a deduction.

$$\frac{\begin{array}{cc}\vdots & \vdots\\ A(s) & s=r\end{array}}{A(r)}\,(A(x))$$

where $A(x)$ is composite.

1.2.2 It is easily seen that the following rules of inference are valid in naive set theory as derived rules:

$\vee$ introduction $(\vee I)$

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{A\vee B} \qquad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A\vee B}$$

$\vee$ elimination $(\vee E)$

$$\frac{\begin{array}{ccc} & \cancel{A} & \cancel{B}\\ \vdots & \vdots & \vdots\\ A\vee B & C & C\end{array}}{C}$$

$\exists$ introduction $(\exists I)$

$$\frac{\begin{array}{c}\vdots\\ A(r)\end{array}}{\exists x A(x)}$$

$\exists$ elimination $(\exists E)$

$$\frac{\begin{array}{cc} & \cancel{A(x)}\\ \vdots & \vdots\\ \exists x A(x) & C\end{array}}{C}$$

where x must not occur free in C or in any open assumption other than $A(x)$ in the deduction

$$\begin{array}{c} A(x) \\ \vdots \\ C \end{array}$$

Major premise in the $\vee E$ inferences will be $A \vee B$ and minor premises the two occurrences of C as premise of the inference. $\exists x A(x)$ is major premise in the $\exists E$ inferences and C the minor premise.

1.3 The formal system of naive set theory with intuitionistic logic.

1.3.1 Add to the rules for forming formulas given in 1.1 the following clauses:
if A, B are formulas, then $A \vee B$ is a formula
if $A(x)$ is a formula, then $\exists x A(x)$ is a formula.

1.3.2 The rules of inference are obtained from 1.2.1. by replacing the $\perp E$ -rule with the following $\perp E$ -rule:

$$\dfrac{\begin{array}{c}\vdots \\ \perp\end{array}}{A}$$

and by adding the derived rules of 1.2.2 as rules of inference.

1.3.3 By using only the formal system based on the rules for $\in, =, \rightarrow, \forall$ we can, in naive set theory, define the other logical constants used here:

$$\perp \underset{\text{def.}}{=} \forall x \forall y\, (x \in y)$$

$$A \& B \underset{\text{def.}}{=} \forall x \forall y\, ((A \rightarrow (B \rightarrow x \in y) \rightarrow x \in y)$$

$$A \vee B \underset{\text{def.}}{=} \forall x \forall y\, (((A \rightarrow x \in y) \& (B \rightarrow x \in y)) \rightarrow x \in y)$$

$$\exists z A(z) \underset{\text{def.}}{=} \forall x \forall y\, (\forall z\, (A(z) \rightarrow x \in y) \rightarrow x \in y)$$

It is now easy to see that the rules for $\bot$, $\&$, $\vee$ and $\exists$ are provable as derived rules in the restricted system. As examples we may look at the $\bot$ and $\exists$-cases:

1.3.3.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\forall x \forall y\,(x \epsilon y)}\,\forall E}{\forall y\,(x \epsilon y)}\,\forall E}{x \epsilon \{z \mid A\}}\,\epsilon E}{A}$$

So $\dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{A}$ is valid as a derived rule.

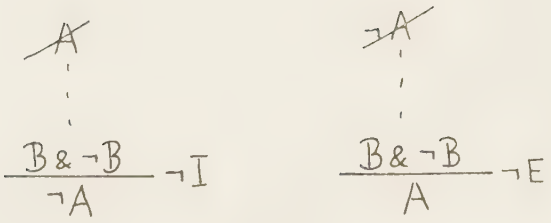
1.3.3.2

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A(r)\end{array} \quad \dfrac{\forall z(A(z) \to x \epsilon y)}{A(r) \to x \epsilon y}\,\forall E}{x \epsilon y}\,\to E}{\forall z\,(A(z) \to x \epsilon y) \to x \epsilon y}\,\to I}{\forall y\,(\forall z\,(A(z) \to x \epsilon y) \to x \epsilon y)}\,\forall I}{\forall x \forall y\,(\forall z\,(A(z) \to x \epsilon y) \to x \epsilon y)}\,\forall I$$

So $\dfrac{\begin{array}{c}\vdots\\ A(r)\end{array}}{\exists z\, A(z)}$ is valid as a derived rule.

1.3.3.3

$$\dfrac{\dfrac{\dfrac{



But in view of the basic nature of the $\longrightarrow$-rules in a system of natural deduction it seems more natural to represent negation in terms of $\longrightarrow$ and $\bot$.

1.5 Notes

For a treatment of the basic concepts of natural deduction the reader is referred to Gentzen ([G]), Prawitz ([Pr1], [Pr2]) and Martin-Löf ([M1]).

2 Inversion and reduction properties of the rules of inference

2.1 By simple operations on $\mathcal{D}$ using $\mathcal{D}_1 \ldots \mathcal{D}_n$ and terms $t_1 \ldots t_m$ I mean substitution of terms t_i for free variables in $\mathcal{D}$, at occurrences where the variables are free for substitution, substitution of deductions $\mathcal{D}_j$ for open assumptions in $\mathcal{D}$ which are conclusions of $\mathcal{D}_j$ and the operation of just leaving $\mathcal{D}$ as it is.

Given two rules of inference $\bar{\Phi}$ and $\bar{\Psi}$, $\bar{\Psi}$ is said to be an inversion of $\bar{\Phi}$ if there are deductions

$$\left.\dfrac{\mathcal{D}_1' \ldots \mathcal{D}_n' \qquad \dfrac{\mathcal{D}_1^2 \ldots \mathcal{D}_m^2}{A}\,\bar{\Phi}}{B}\,\bar{\Psi}\right\}\mathcal{D}$$

where A is the major premise of the inference if $\bar{\Psi}$ is an E-inference, and for all such deductions we can obtain uniformly a deduction $\mathcal{D}^*$ of B from the open assumptions in $\mathcal{D}$ by iterated simple operations on $\mathcal{D}_{i\le m}^2$ $(\mathcal{D}_{j\le n}')$ using $\mathcal{D}_{j\le n}'$ $(\mathcal{D}_{i\le m}^2)$ and terms occurring in B.

2.1.1 $\&E$ is an inversion of $\&I$

Given

$$\dfrac{\dfrac{\begin{matrix}\vdots\\A\end{matrix}\}\mathcal{D}_1 \quad \begin{matrix}\vdots\\B\end{matrix}\}\mathcal{D}_2}{A\&B}\,\&I}{B}\,\&E$$

we just take $\mathcal{D}_2$ and so obtain the desired deduction by a simple operation on $\mathcal{D}_2$.

2.1.2 $\rightarrow E$ is an inversion of $\rightarrow I$

Given

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}\Big\}\mathcal{D}_1 \quad \dfrac{\begin{array}{c}\not{A}\\ \vdots\\ B\end{array}\Big\}\mathcal{D}_2}{A \rightarrow B}\rightarrow I}{B}\rightarrow E$$

we use $\mathcal{D}_1$ to substitute for A in $\mathcal{D}_2$ to obtain

$$\begin{array}{c}\vdots\\ A\\ \vdots\\ B\end{array}$$

by a simple operation on $\mathcal{D}_2$ using $\mathcal{D}_1$.

2.1.3 $\forall E$ is an inversion of $\forall I$.

Given

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A(x)\end{array}\Big\}\mathcal{D}_1}{\forall x\, A(x)}\forall I}{A(t)}\forall E$$

we stubstitute t for x in $\mathcal{D}_1$ to obtain $\mathcal{D}_1(t/x)$
by a simple operation on $\mathcal{D}_1$ using a term in $A(t)$.

2.1.4 ϵE is an inversion of ϵI

Given

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A(r)\end{array}\Big\}\mathcal{D}_1}{r \epsilon \{x \mid A(x)\}}\epsilon I}{A(r)}\epsilon E$$

we just take $\mathcal{D}_1$, this being a simple operation on $\mathcal{D}_1$.

2.1.5 When we look at the $=$-rules we see that we have inversion properties only for very special applications of the rules. Namely given

$$(t\epsilon x)\ \dfrac{\mathcal{D}_1\left\{\begin{array}{c}\vdots\\ t\epsilon S\end{array}\right.\quad \mathcal{D}_2\left\{\begin{array}{cc}\cancel{x\epsilon S} & \cancel{x\epsilon r}\\ \vdots & \vdots\\ x\epsilon r & x\epsilon S\end{array}\right\}\mathcal{D}_3}{t\epsilon r}\ \begin{array}{l}\\ S=r\end{array}\ \begin{array}{l}=I\\ \\ =E\end{array}$$

we substitute t for x in $\mathcal{D}_2$ to obtain $\mathcal{D}_2(t/x)$ and then substitute $\mathcal{D}_1$ for $t\epsilon S$ in $\mathcal{D}_2(t/x)$ to obtain

$$\left.\begin{array}{c}\vdots\\ t\epsilon S\end{array}\right\}\mathcal{D}_1$$
$$\left.\begin{array}{c}\vdots\\ t\epsilon r\end{array}\right\}\mathcal{D}_2(t/x)$$

by iterated simple operations on $\mathcal{D}_2$ using t and $\mathcal{D}_1$.

What we would like to have is a certain reduction of the other cases to this special case and thereby have a basis case with inversion properties to which the other cases are reducable. Unfortunately not all basis cases have the desirable inversion properties:

$$(t(x)\epsilon x)\ \dfrac{\mathcal{D}_1\left\{\begin{array}{c}\vdots\\ t(s)\epsilon S\end{array}\right.\quad \mathcal{D}_2\left\{\begin{array}{cc}\cancel{x\epsilon S} & \cancel{x\epsilon r}\\ \vdots & \vdots\\ x\epsilon r & x\epsilon S\end{array}\right\}\mathcal{D}_3}{t(r)\epsilon r}\ \begin{array}{l}\\ S=r\end{array}$$

By iterated simple operations on $\mathcal{D}_2$ using $\mathcal{D}_1$ and terms in $t(s)\epsilon S$ we can only obtain

$$\left.\begin{array}{c}\vdots\\ t(s)\epsilon S\end{array}\right\}\mathcal{D}_1$$
$$\left.\begin{array}{c}\vdots\\ t(s)\epsilon r\end{array}\right\}\mathcal{D}_2\left(t(s)/x\right)$$

so we have to add an application of the $=E$ rule to get

$$\dfrac{\begin{array}{c}\vdots\\ t(s)\in s\\ \vdots\end{array}\qquad\qquad\begin{array}{c}\vdots\\ s=r\end{array}}{(t(x)\in r)\quad t(s)\in r\qquad\qquad t(r)\in r}\ =E$$

and if r is an abstraction term this $=E$ inference is no longer a basis case, so one would again like to have a reduction to a basis case having the inversion properties we are looking for.

2.2 Let Φ be an E-rule. I will say that Φ is <u>reducible</u> if for all composite applications

$$\left.\dfrac{\mathcal{D}_1 \ldots \mathcal{D}_n \qquad \mathcal{D}}{\mathcal{B}}\ \Phi\right\}\mathcal{D}'$$

where $\mathcal{D}$ is the major deduction of the inference, we can uniformly obtain a deduction $\mathcal{D}^*$ of B from the open assumptions of $\mathcal{D}'$ on I-form such that new applications relative to $\mathcal{D}'$ of the Φ-rule in $\mathcal{D}^*$ have the form

$$\dfrac{\mathcal{D}_1' \ldots \mathcal{D}_m' \quad A \quad \mathcal{D}}{A'}\ \Phi$$ where A is

an assumption cancelled by the last inference in $\mathcal{D}^*$ or A' is a premise in the last inference of $\mathcal{D}^*$.

So if Φ is reducible all composite applications can be reduced to applications to assumptions and premises in the I-rule corresponding to the given composite formula.

2.2.1 $=E$ is reducible.

Let us look at the $\forall$ and $\rightarrow$-cases. The other cases are treated in a similar manner.

Given

$$\left(\forall z A(zx)\right)\frac{\left.\begin{array}{c}\vdots\\ \forall z A(z,s)\end{array}\right\}\mathcal{D}_1 \qquad \left.\begin{array}{c}\vdots\\ s=r\end{array}\right\}\mathcal{D}_2}{\forall z\, A(z,r)}$$

we use $\mathcal{D}_1$ to obtain $\dfrac{\mathcal{D}_1}{A(z,s)}\forall E$

and adding $\mathcal{D}_2$ we have

$$\frac{\dfrac{\mathcal{D}_1}{A(z,s)} \quad \mathcal{D}_2}{A(z,r)}=E$$

so we then get

$$\frac{\dfrac{\dfrac{\mathcal{D}_1}{A(z,s)} \quad \mathcal{D}_2}{A(z,r)}\forall 1}{\forall z A(z,r)}$$

Given

$$\left(A(x)\rightarrow B(x)\right)\frac{\left.\begin{array}{c}\vdots\\ A(s)\rightarrow B(s)\end{array}\right\}\mathcal{D}_1 \qquad \left.\begin{array}{c}\vdots\\ s=r\end{array}\right\}\mathcal{D}_2}{A(r)\rightarrow B(r)}$$

we use $\mathcal{D}_2$ to obtain $\dfrac{(A(x))\quad A(r)\quad \mathcal{D}_2}{A(s)} = E$

so using this deduction we obtain

$$\dfrac{\dfrac{(A(x))\quad A(r)\quad \mathcal{D}_2}{A(s)}\quad \mathcal{D}_1}{B(s)} \to E$$

Using $\mathcal{D}_2$ once again we obtain

$$\dfrac{\dfrac{(A(x))\quad A(r)\quad \mathcal{D}_2}{A(s)}\quad \mathcal{D}_1}{\dfrac{(B(x))\quad B(s)\quad \mathcal{D}_2}{B(r)} = E}$$

and so finally we have

$$\dfrac{(A(x))\quad \cancel{A(r)}\quad \mathcal{D}_2}{\dfrac{A(s)\quad \mathcal{D}_1}{\dfrac{(B(x))\quad B(s)\quad \mathcal{D}_2}{B(r)}}} \to I$$

$$A(r) \to B(r)$$

2.2.2 $\perp E$ is reducible.

As in 2.2.1 all cases are treated in a similar manner, so let us look at the $\to$ case.

Given

$$\dfrac{\cancel{\neg(A \to B)}\quad \vdots\quad \perp}{A \to B}$$

use

$$\dfrac{\dfrac{A \quad A \to B}{B} \qquad \neg B}{\bot}$$

to obtain

$$\dfrac{\dfrac{\dfrac{A \quad \cancel{A \to B}}{B} \qquad \neg B}{\bot}}{\neg(A \to B)} \to I$$

and we have

$$\dfrac{\dfrac{\dfrac{\dfrac{A \quad \cancel{A \to B}}{B} \qquad \cancel{\neg B}}{\bot}}{\neg(A \to B)}}{\vdots}$$
$$\dfrac{\bot}{B} \bot E$$

and finally

$$\dfrac{\dfrac{\dfrac{\dfrac{\cancel{A} \quad \cancel{A \to B}}{B} \qquad \cancel{\neg B}}{\bot}}{\neg(A \to B)}}{\vdots}$$
$$\dfrac{\dfrac{\bot}{B}}{A \to B} \to I$$

2.3 Inversion and reduction properties of the rules of inference in naive set theory with intuitionistic logic

Reasoning as in 2.1 it is easily seen that $\vee E$ is an inversion of $\vee I$ and $\exists E$ an inversion of $\exists I$.

2.4 Notes

The idea of giving an explicit formulation of the inversion properties of pairs of rules goes back to Prawitz. (See [Pr1] chapter II pp 32-38).

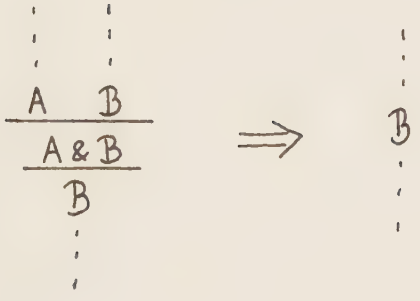
3. Rules of contraction

An occurrence of a composite formula A in a deduction $\mathfrak{D}$ which is the conclusion of an $\perp$ E-inference or both the conclusion of an I-inference and the major premise of an E-inference is called a cut in $\mathfrak{D}$.

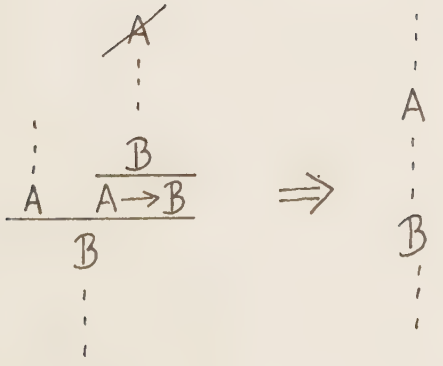
A natural way of expressing the inversion and reduction properties discussed in section 2 is in terms of certain rules of contraction which give us instructions how to eliminate or change the context of a cut in a deduction $\mathfrak{D}(\Gamma; A)$ and thereby obtain a new deduction $\mathfrak{D}'(\Gamma; A)$ called the contraction of $\mathfrak{D}$ at this cut.

3.1 Rules of contraction for naive set theory

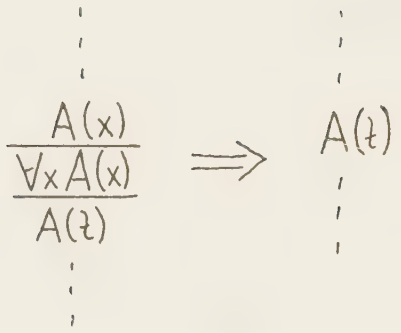
3.1.1 &



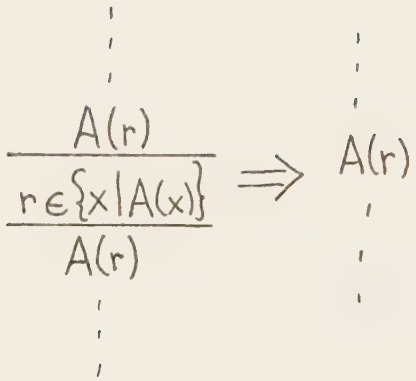
3.1.2 →



3.1.3 $\forall$



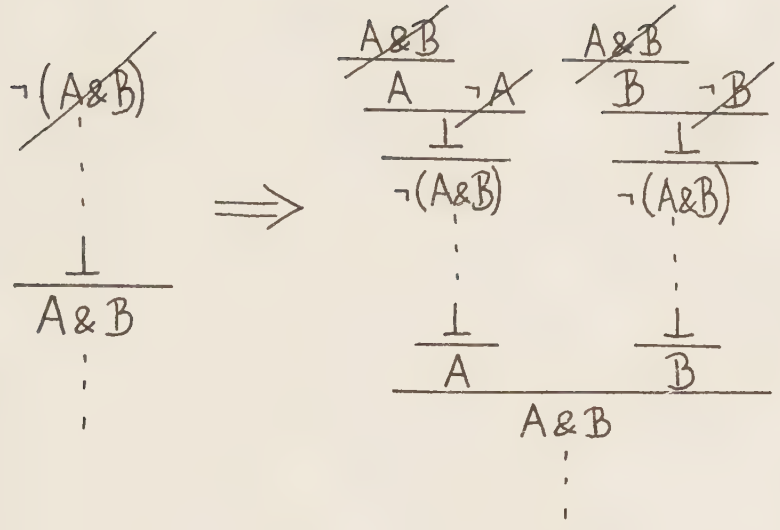
3.1.4 $\in$



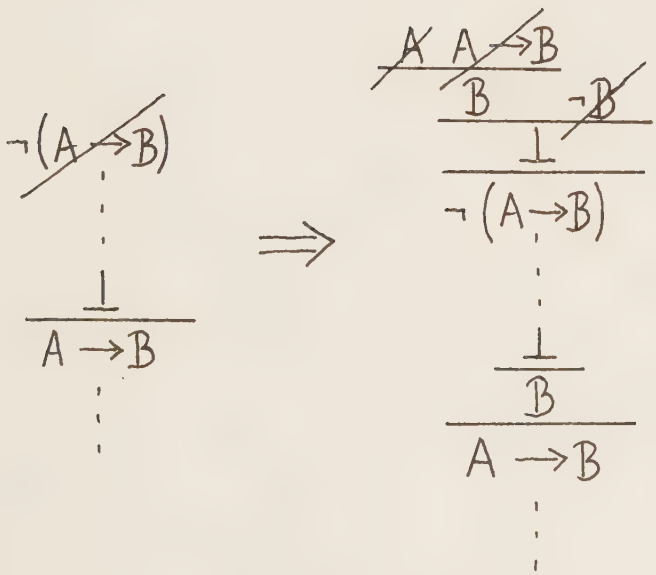
3.1.5 $\perp E$

These rules will be dependent on the form of the cut A, so there are several cases.

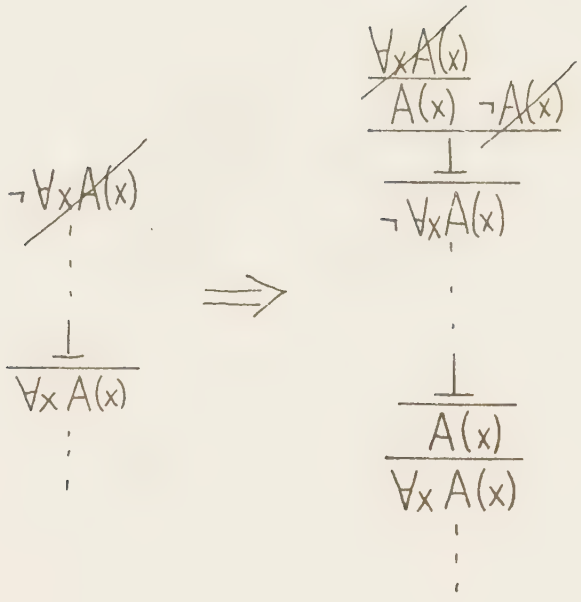
3.1.5.1 &



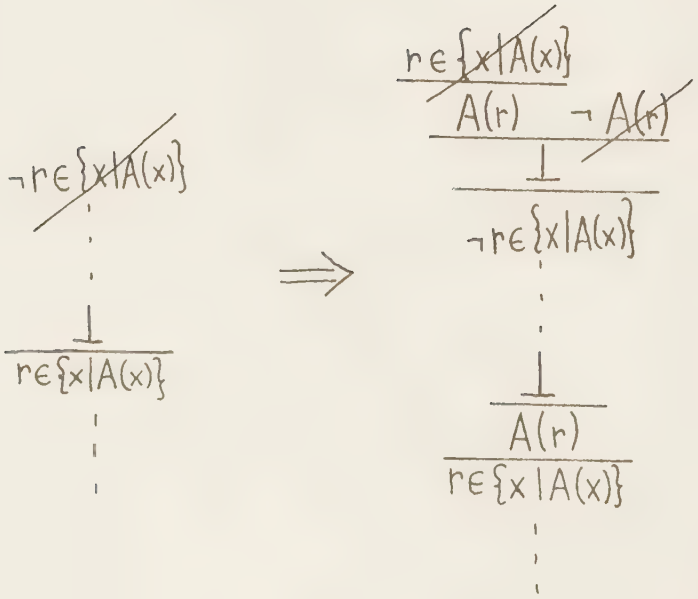
3.1.5.2 →



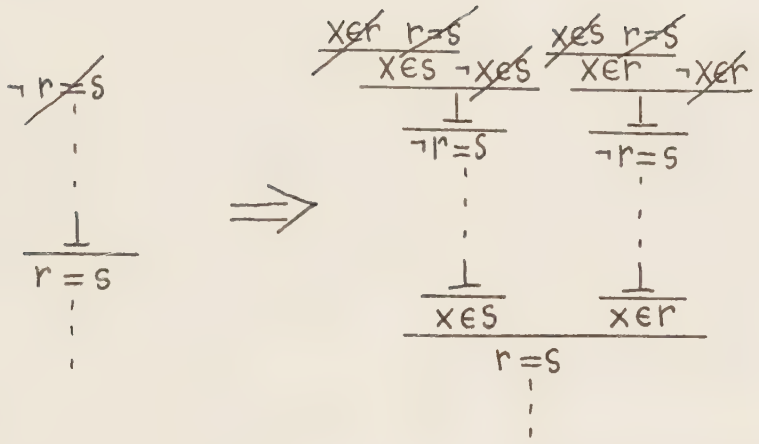
3.1.5.3 $\forall$



3.1.5.4 $\in$



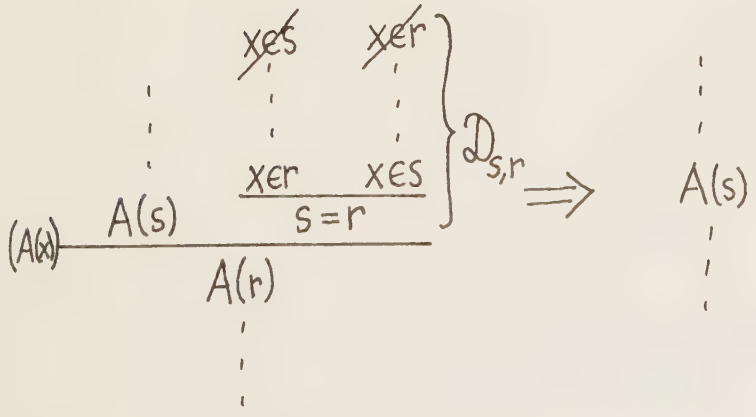
3.1.5.5 =



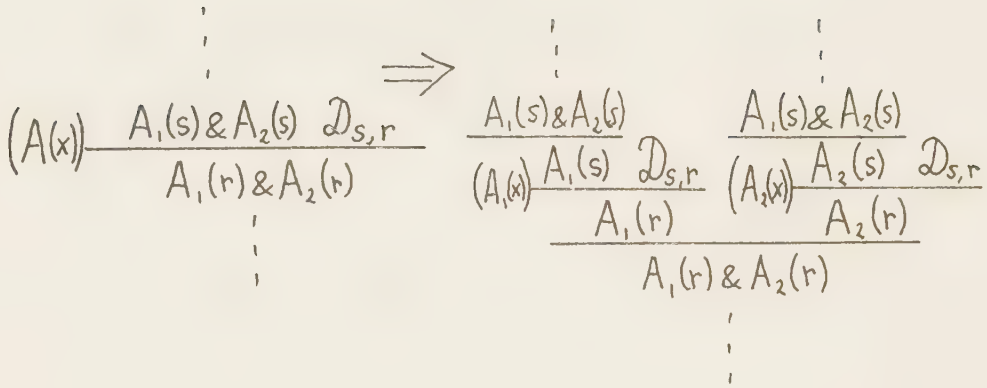
3.1.6 =

The rules will here depend on the form of A(x) so we have to distinguish several cases.

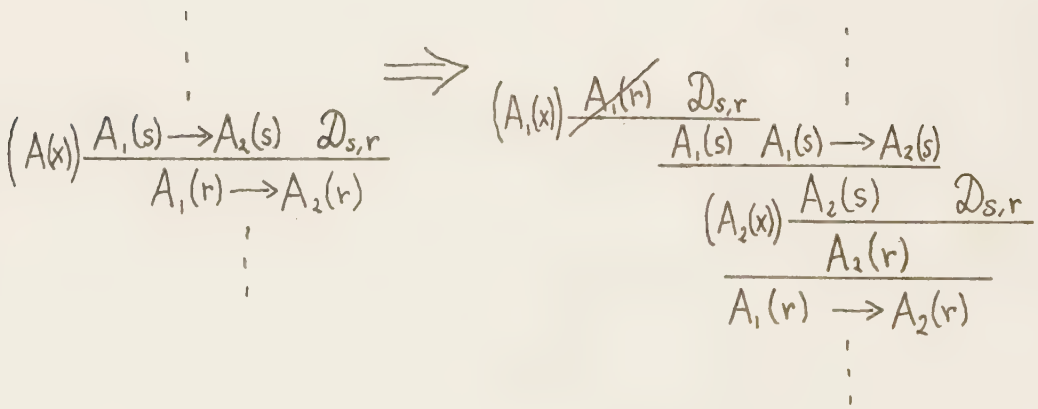
3.1.6.1 x is not free in A(x)



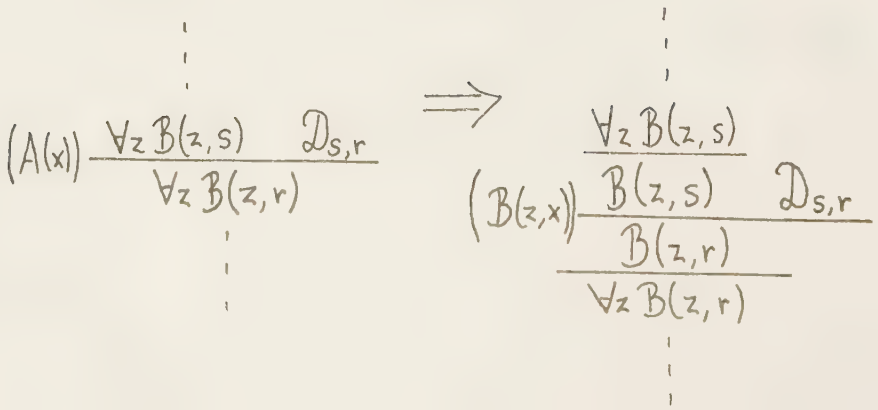
3.1.6.2 $A(x) \equiv A_1(x) \& A_2(x)$



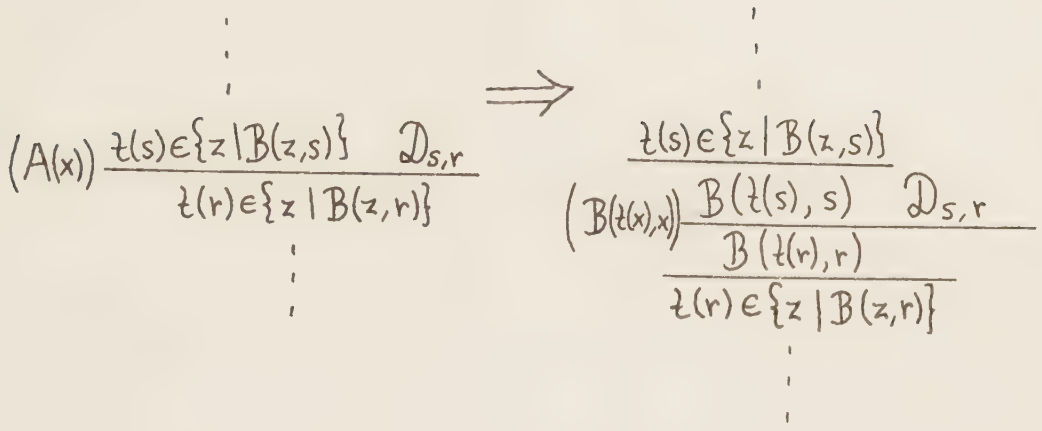
3.1.6.3 $A(x) \equiv A_1(x) \rightarrow A_2(x)$



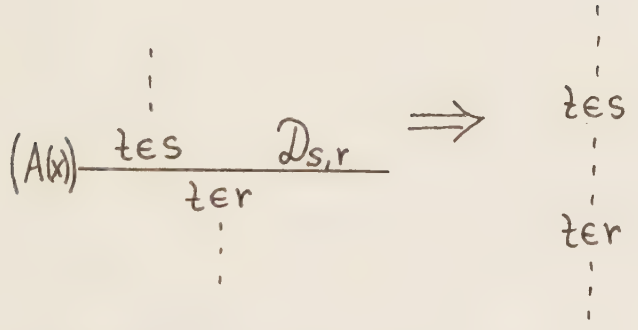
3.1.6.4 $A(x) \equiv \forall z\, B(z,x)$



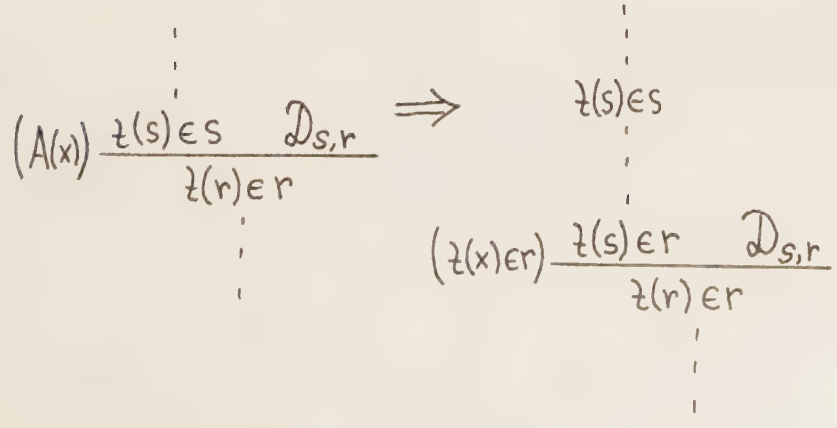
3.1.6.5 $A(x) \equiv t(x) \in \{z \mid B(z,x)\}$



3.1.6.6 $A(x) \equiv t \in x$ (x not free in t)



3.1.6.7 $A(x) \equiv t(x) \in x$ (x not free in t)



3.1.6.8 $A(x) \equiv t(x) \in z$

Given

$$\dfrac{\begin{array}{c}\vdots\\ t(s)\in z\end{array} \quad \mathcal{D}_{s,r}}{\begin{array}{c}t(r)\in z\\ \vdots\end{array}}$$

Since $t(s)\in z$ is atomic and the substitution of r for s only takes place in t(x) we have here a case where there are no inversion or reduction properties to explicate in terms of a rule of contraction. We thus leave this cut unanalyzed and define no rule of contraction here.

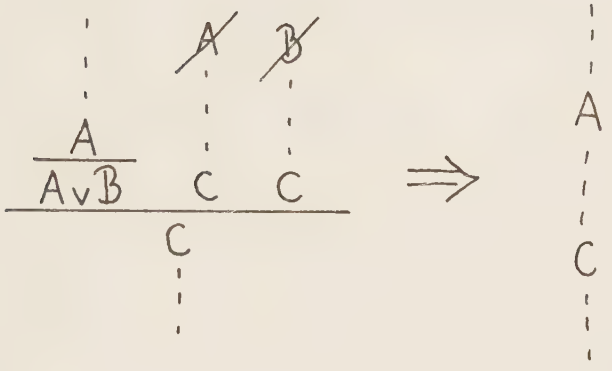
3.1.6.9 $A(x) \equiv t(x) = u(x)$

$$(A(x))\ \dfrac{\begin{array}{c}\vdots\\ t(s)=u(s)\end{array} \quad \mathcal{D}_{s,r}}{\begin{array}{c}t(r)=u(r)\\ \vdots\end{array}} \quad \Longrightarrow$$

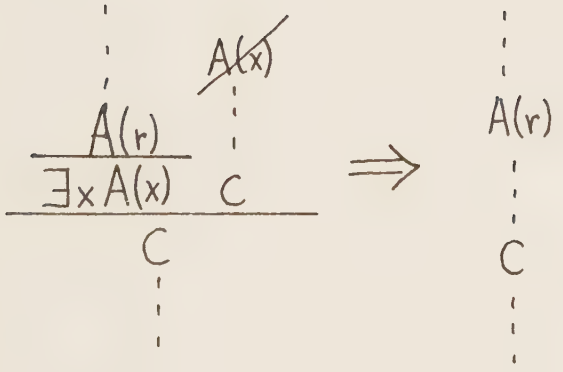
$$\Longrightarrow \quad \dfrac{\dfrac{(z\in t(x))\ \dfrac{z\in t(r) \quad \mathcal{D}_{s,r}}{z\in u(r)}}{(z\in x)\ \dfrac{z\in t(s) \quad t(s)=u(s)}{z\in u(s)}} \qquad \dfrac{(z\in u(x))\ \dfrac{z\in u(r) \quad \mathcal{D}_{s,r}}{z\in t(r)}}{(z\in x)\ \dfrac{z\in u(s) \quad t(s)=u(s)}{(z\in t(x))\ \dfrac{z\in t(s) \quad \mathcal{D}_{s,r}}{z\in t(r)}}}}{\begin{array}{c}t(r)=u(r)\\ \vdots\end{array}}$$

3.2 To obtain rules of contraction for naive set theory with intuitionistic logic we omit all rules concerning $\bot$ E-inferences and add the following rules of contraction for $\vee$, and $\exists$:

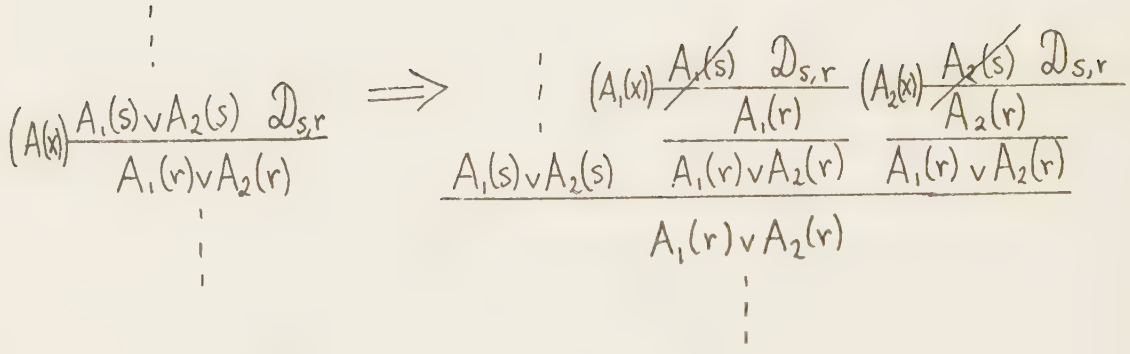
3.2.1 $\vee$



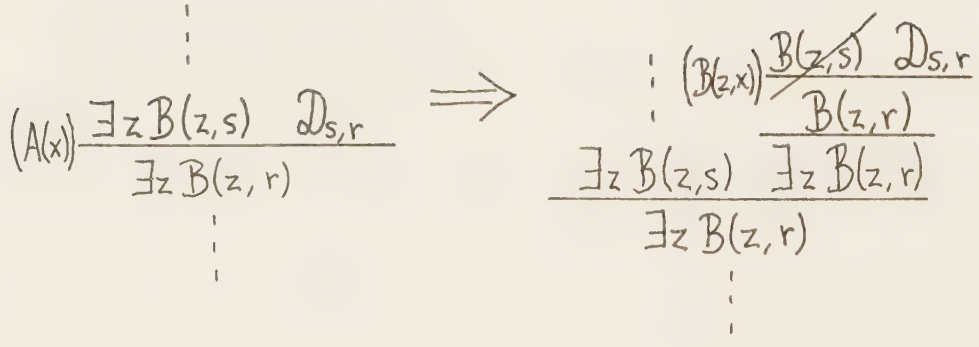
3.2.2 $\exists$



3.2.3 $=$ $A(x) \equiv A_1(x) \vee A_2(x)$



3.2.4 $=$ $A(x) \equiv \exists z B(z,x)$



3.3 Notes

The $\&, \vee, \rightarrow, \forall, \exists$ and $\perp$ E-rules of contraction given in this section follows Prawitz. (See [Pr 1] pp 35-40, 94-95).

4 Normalizability

Given the rules of contraction as an explication of the inversion and reduction properties of the rules of inference, we are of course interested in what this explication tells us about these properties. A natural question is then to ask whether sequences of contractions are well-founded or not. The more precise content of this question is given by the concept of normalizability.

4.1 Main cut and main branch

If $\mathcal{D}$ is on E-form we may follow major premises of E-inferences in $\mathcal{D}$ upwards until we reach a cut or an assumption.
If we reach a cut and there is defined a contraction at this cut it will be called the main cut in $\mathcal{D}$.
If there is no contraction defined, the minor premise is atomic and so either an assumption or the conclusion of an E-inference. In this case we go to the minor premise instead of the major premise and continue upwards in $\mathcal{D}$ following major premises of E-inferences until we perhaps reach a new cut which will be the main cut in $\mathcal{D}$ if there is a contraction defined; otherwise we go to the minor premise and follow major premises of E-inferences upwards in $\mathcal{D}$ etc... The branch in $\mathcal{D}$ followed in this manner will be called the main branch of $\mathcal{D}$.

4.2 Normal deductions

If $\mathcal{D}$ consists only of an assumption, then $\mathcal{D}$ is normal.
If $\mathcal{D}$ is on I-form, then $\mathcal{D}$ is normal if the premise deductions in the last inference of $\mathcal{D}$ are all normal.
If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ is normal if $\mathcal{D}$ has a cut free main branch with normal minor and major deductions along it.

4.3 Normalizable deductions

If $\mathcal{D}$ consists only of an assumption, then $\mathcal{D}$ is normalizable.
If $\mathcal{D}$ is on I-form, then $\mathcal{D}$ is normalizable if the premise deductions of the last inference of $\mathcal{D}$ are all normalizable.
If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ is normalizable if $\mathcal{D}$ has a cut-free main branch with normalizable minor and major deductions along it or $\mathcal{D}$ has a main cut and the contraction of $\mathcal{D}$ at this cut is normalizable.

4.4 Properties of normalizable deductions

I will say that $\mathcal{D}$ reduces to a normal deduction $\mathcal{D}'$ if we can obtain $\mathcal{D}'$ from $\mathcal{D}$ by repeatedly applying the rules of contraction starting with some cut in $\mathcal{D}$ if there are any.

4.4.1 By induction on the definition of " $\mathcal{D}$ is normalizable" it is immediate that if $\mathcal{D}(\Gamma; A)$ is normalizable, then $\mathcal{D}$ reduces to a normal deduction $\mathcal{D}'(\Gamma; A)$.

4.4.2 Any closed normal deduction $\mathcal{D}$ is on I-form.

Assume $\mathcal{D}$ is on E-form.
Since no contractions are defined for atomic applications of the $\perp$ E-rule, there may be some assumptions cancelled along the main branch of $\mathcal{D}$ of the form $\neg rex$, but certainly no other assumptions are cancelled along the main branch in $\mathcal{D}$. So following the main branch in $\mathcal{D}$ upwards we will end up with an assumption. If it is not of the form $\neg rex$ we know immediately that $\mathcal{D}$ is open, so assume our assumption is of the form $\neg rex$.
We know $\neg rex$ is the major premise of an E-inference:

$$\mathcal{D}'\left\{\begin{array}{c}\vdots\\ rex \quad \neg rex\end{array}\right.$$
$$\perp$$
$$\vdots$$

This follows from the fact that the only minor premises of E-inferences we may find along the main branch are atomic minor premises of $=$E-inferences.

Now $\mathcal{D}'$ is on E-form, so we may follow the main branch in $\mathcal{D}'$ upwards until we end up with an assumption of the form $\neg u \in z$ or of some other form.
Reasoning as above we may continue in this manner and finally we have to find an assumption not cancelled in $\mathcal{D}$, so $\mathcal{D}$ is open contradicting our basic assumption. So if $\mathcal{D}$ is closed and normal, then $\mathcal{D}$ is on I-form.

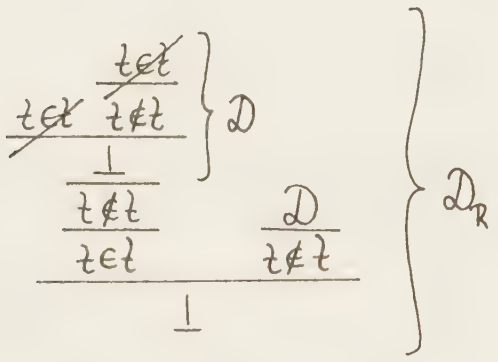
4.5 Notes

The particular definition of normalizability used here is taken from Martin-Löf. (See [M1] pp 195-199).

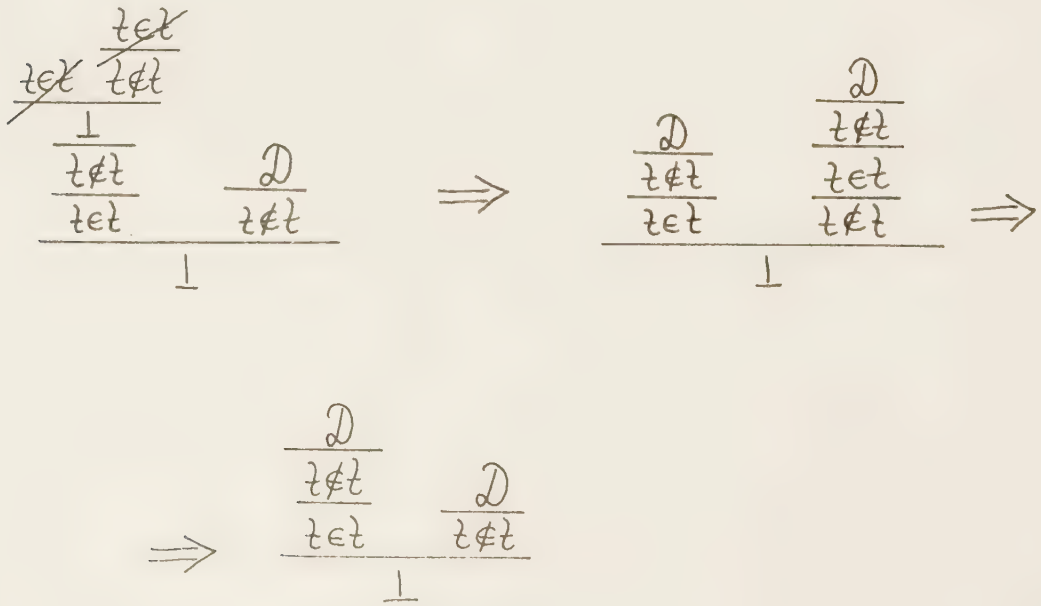
5 Counter examples to normalizability in set theory

5.1 Russell's paradox

Russell's paradox shows that naive set theory is inconsistent and gives at the same time a very simple example of a deduction in naive set theory which is not normalizable.
Let t be the term $\{x \mid x \notin x\}$:



Now since $\mathcal{D}_R$ ends with $\bot$ and is closed we know already by 4.4.1 and 4.4.2 that $\mathcal{D}_R$ cannot be normalizable. And if we try to reduce $\mathcal{D}_R$ to a normal deduction following the definition of normalizability we get the following chain of deductions:



So we see that after two contractions we are back at the deduction we started with. The reason for this lies in the fact that $t \in t$ so to speak both contains $t \notin t$ and is contained in $t \notin t$ in terms of the rules of inference:

$$\frac{t \notin t}{t \in t}\in I \qquad \frac{t \in t}{t \notin t}\in E \qquad \frac{\dfrac{\dfrac{t \in t}{\bot}\in E \quad t \notin t}{\bot}\to E}{t \notin t}\to I$$

5.2 Crabbe's counter example

By extending Russell's paradox, Marcel Crabbe gave an example of a closed deduction on I-form which is not normalizable using the $\in$-rules only in the form of Zermelo's subset axiom.

Let u be a term different from x and t the term $\{x \mid x \in u \& x \notin x\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t \in t}{t \in u \& t \notin t}}{\bot}\,\mathcal{D}}{t \in u \quad t \notin t}}{t \in u \& t \notin t}\quad \mathcal{D}}{t \in t \quad t \notin t}}{\bot}}{t \notin u}\;\mathcal{D}_C$$

If we try to reduce $\mathcal{D}_C$ to a normal deduction by following the definition of normalizability we will be back at $\mathcal{D}_C$ again after three steps. The reasons for this are basically the same as in the case of Russell's paradox.

In set theoretical practice we encounter deductions of the same form as $\mathcal{D}_C$ in arguments showing that certain classes are not sets in some given system of set theory. As a typical example we have the argument showing that $\{x|x=x\}$ is not a set in Zermelo set theory:
Assume $\{x|x=x\}$ is a set. Then by the subset axiom $\{z \mid z\in\{x\mid x=x\}\ \&\ z\notin z\}$ is a set. Let us call it M. Now $M\notin M$ is true and so is $M\in\{x|x=x\}$, so $M\in M$ which is a contradiction, so $\{x|x=x\}$ is after all not a set.
But as an argument for $t\notin u$ this type of argument is a rather strange one, very indirect and not very informative. A more natural proof of $t\notin u$ is easily obtained using a rule for set induction:

$$\dfrac{\begin{array}{c}\forall x\in y A(x)\\ \vdots\\ A(y)\end{array}}{A(t)}\ \text{Set. ind.}$$

y not free in any other open assumption than $\forall x\in y\, A(x)$ in the deduction

$$\begin{array}{c}\forall x\in y A(x)\\ \vdots\\ A(y)\end{array}$$

$$\dfrac{\mathcal{D}\left\{\begin{array}{l}\dfrac{\dfrac{\dfrac{\forall y\in x(y\notin y)}{x\in x\to x\notin x}\quad x\in x}{\bot}}{x\notin x}\ \text{Set. ind.}\\[2ex] \dfrac{\dfrac{\mathcal{D}}{x\notin x}\quad x\in u}{x\in u\ \&\ x\notin x}\\[2ex] \dfrac{x\in t}{x\in u\ \&\ x\notin x}\ \text{Set. ind.}\end{array}\right.}{\dfrac{t\notin t}{u=t}\ (t\notin x)}$$

$$t\notin u$$

So this proof tells us that the reason why $t \notin u$ is that sets are not members of themselves, which means that t and u are equal and so t is then not a member of u.
To be able to say something about the reduction of this deduction to a normal deduction we must first have an analysis of set induction in terms of rules of contraction. I have no idea of what such an analysis of the general rule of set induction should look like.

5.3 A counter example involving = -rules

5.3.1 Atomic applications of the =E-rule.

Let t be a given term and let u be the term $\{x \mid x \in x\}$.
Let be a deduction

$$\left.\begin{array}{c} \not{x \in u} \quad \not{x \in t} \\ \vdots \quad\quad \vdots \\ \dfrac{x \in t \quad\quad x \in u}{u = t} \end{array}\right\} \mathcal{D}_{u,t}$$

of $u=t$ on I-form.
Now if $\begin{array}{c} \vdots \\ t \in t \end{array}$ is any deduction of $t \in t$
consider the following deduction:

$$(x \in x)\dfrac{\begin{array}{c}\vdots \\ t \in t\end{array} \quad \mathcal{D}_{u,t}}{u \in u}$$

If we try to reduce this deduction to a normal deduction following the definition of normalizability we will get the following infinite chain of deductions:

$$(x \in x)\dfrac{\begin{array}{c}\vdots \\ t \in t\end{array} \quad \mathcal{D}_{u,t}}{u \in u} \Longrightarrow (x \in u)\dfrac{\left.\begin{array}{c}\vdots \\ t \in t \\ \vdots \\ t \in u\end{array}\right\}\mathcal{D}_1 \quad \mathcal{D}_{u,t}}{u \in u} \Longrightarrow$$

$$\Rightarrow (x\in X)\frac{\dfrac{\mathcal{D}_1}{t\in t}}{\dfrac{u\in u}{u\in u}}\ \mathcal{D}_{u,t} \quad\Rightarrow\quad (x\in u)\frac{\left.\begin{array}{c}\dfrac{\mathcal{D}_1}{t\in t}\\ \vdots\end{array}\right\}\mathcal{D}_2}{\dfrac{t\in u}{u\in u}}\ \mathcal{D}_{u,t} \Rightarrow$$

$$\Rightarrow (x\in X)\frac{\dfrac{\dfrac{\mathcal{D}_2}{t\in t}}{u\in u}\ \mathcal{D}_{u,t}}{\dfrac{u\in u}{u\in u}} \Rightarrow$$

etc...

It is easily seen that instead of $\{x \mid x\in X\}$ we could have used $\{x \mid x\in r \,\&\, x\in X\}$ for a given term r, and the reduction sequence would not have stopped in this case either. So the counter example is reproducible in all ordinary fragments of naive set theory.

5.3.2 Composite applications of the $=$E-rule

Given terms s and u, and a deduction $\mathcal{D}_{s,u}(\Gamma; s=u)$ on I-form let $t(s)$ be $\{x \mid x\in s \,\&\, x\in X\}$.

Consider the following deduction

$$(t(x)\in t(x))\frac{t(s)\in t(s)\quad \mathcal{D}_{s,u}}{t(u)\in t(u)}$$

If we try to reduce this deduction to a normal deduction following the definition of normalizability we will get a chain of contractions as follows:

$$\left(t(x)\in t(x)\right)\dfrac{t(s)\in t(s)\quad \mathcal{D}_{s,u}}{t(u)\in t(u)}\quad\Longrightarrow$$

$$\left(t(x)\in x \,\&\, t(x)\in t(x)\right)\dfrac{\dfrac{t(s)\in t(s)}{t(s)\in s \,\&\, t(s)\in t(s)}\quad \mathcal{D}_{s,u}}{\dfrac{t(u)\in u \,\&\, t(u)\in t(u)}{t(u)\in t(u)}}\quad\Longrightarrow$$

$$\Longrightarrow$$

$$\left(t(x)\in x\right)\dfrac{\dfrac{\dfrac{t(s)\in t(s)}{t(s)\in s \,\&\, t(s)\in t(s)}}{t(s)\in s}\ \mathcal{D}_{s,u}\quad *\dfrac{\dfrac{\dfrac{t(s)\in t(s)}{t(s)\in s \,\&\, t(s)\in t(s)}}{t(s)\in t(s)}\ \mathcal{D}_{s,u}}{t(u)\in t(u)}\quad\Longrightarrow$$

$$\Longrightarrow\qquad *\left(t(x)\in t(x)\right)$$

$$\dfrac{t(u)\in u}{\dfrac{t(u)\in u \,\&\, t(u)\in t(u)}{t(u)\in t(u)}}$$

$$\Longrightarrow\quad \text{etc...}$$

5.4 A counter example involving the $\bot$ E-rule

Let u be the term $\{x \mid x \in x\}$.
Given a deduction

$$\begin{array}{c} \neg u \in u \\ \vdots \\ \bot \end{array}$$

we form the deduction

$$\left.\begin{array}{c} \cancel{\neg u \in u} \\ \vdots \\ \dfrac{\bot}{u \in u} \end{array}\right\} \mathcal{D}$$

If we try to normalize $\mathcal{D}$ we will get the following chain of contractions:

$$\begin{array}{c} \cancel{\neg u \in u} \\ \vdots \\ \dfrac{\bot}{u \in u} \end{array} \Rightarrow \begin{array}{c} \dfrac{\cancel{u \in u}}{\dfrac{u \in u \quad \cancel{\neg u \in u}}{\bot}} \\ \neg u \in u \\ \vdots \\ \dfrac{\bot}{\dfrac{u \in u}{u \in u}} \end{array} \Rightarrow$$

$$\Rightarrow \begin{array}{c} \dfrac{\cancel{u \in u}}{\dfrac{u \in u \quad \cancel{\neg u \in u}}{\dfrac{\cancel{u \in u} \quad \bot}{\dfrac{u \in u \quad \neg u \in u}{\bot}}}} \\ \neg u \in u \\ \vdots \\ \dfrac{\bot}{\dfrac{u \in u}{\dfrac{u \in u}{u \in u}}} \end{array} \Rightarrow \text{etc...}$$

It is easily seen that we will get a counter example to normalizability if we let u be the term $\{x \mid x \in r \& x \in x\}$, for a given term r, by just extending the given counter example in the obvious manner.

The counter examples given here in section 5 make it quite clear that the inversion and reduction properties of the rules of inference found by purely syntactical means are not sufficient for ensuring normalizability.

5.5 Notes

The use of Russell's paradox as a counter example to normalizability is taken from Prawitz. (See [Pr] pp 94-95.)

6 $\mathcal{C}$-normalizability

In view of the counter examples to normalizability given in the previous section, there are basically two different kinds of restrictions one could impose:

(i) one could restrict the study to certain fragments of naive set theory;
(ii) one could look for a more restricted concept of normalizability than the one given in section 3.

If we want a closed normal deduction to be on I-form, Russell's paradox shows that we have to make some restrictions of type (i). And if we shall be able to give an analysis of fragments large enough to contain Zermelo's subset axiom, Crabbe's counter example shows that we also have to introduce some restrictions of type (ii). One such possible restriction on the concept of normalizability will be given here in terms of $\mathcal{C}$-normalizability.

6.1 $\mathcal{C}$-main cut and $\mathcal{C}$-main branch

Let $\mathcal{C}$ be a predicate defined for deductions in naive set theory.

Now I will assume that contractions are defined for certain cuts only when the deduction of the cut and the minor deductions at the cut, if any, satisfy $\mathcal{C}$.

If $\mathcal{D}$ is a deduction on E-form, I will describe a way of proceeding upwards in $\mathcal{D}$ from the conclusion by giving instructions how to go from a given formula to the next, starting with the conclusion:

(i) If the given formula is the conclusion of an E-inference and not a cut, then we proceed upwards to the major premise of the E-inference.

(ii) If the given formula is a cut for which there is a rule of contraction defined, then we stop, and the given formula will be called the $\mathcal{C}$-main cut.

(iii) If the given formula A is a cut for which there is no rule of contraction defined, there are three possibilities:

(a) A is the conclusion of an $\rightarrow$I or =I-inference, Then we go to the minor premise instead.

(b) A is the conclusion of an & I, $\in$ I or $\forall$I-inference. Then we stop, and the deduction of the given formula will be called the $\mathcal{C}$-main deduction.

(c) A is the conclusion of an $\perp$ E-inference, Then we proceed one step upwards in the deduction.

(iv) If the given formula is the conclusion of an I-inference and not a cut, then we stop, and the deduction of the given formula will be called the $\mathcal{C}$-main deduction.

(v) If A is an assumption, then we stop.

The branch in $\mathcal{D}$ followed in this way will be called the $\mathcal{C}$-main branch of $\mathcal{D}$.

6.2 $\mathcal{C}$-normal deductions

If $\mathcal{D}$ consists of an assumption, then $\mathcal{D}$ is $\mathcal{C}$-normal.
If $\mathcal{D}$ is on I-form, then $\mathcal{D}$ is $\mathcal{C}$-normal if the premise deductions of the last inference in $\mathcal{D}$ are all $\mathcal{C}$-normal.
If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ is $\mathcal{C}$-normal if $\mathcal{D}$ has no $\mathcal{C}$-main cut and all minor and major deductions along the $\mathcal{C}$-main branch are $\mathcal{C}$-normal and the $\mathcal{C}$-main deduction, if there is one, is also $\mathcal{C}$-normal.

6.3 $\mathcal{C}$-normalizable deductions

If $\mathcal{D}$ consists only of an assumption, then $\mathcal{D}$ is $\mathcal{C}$-normalizable.

If $\mathcal{D}$ is on I-form, then $\mathcal{D}$ is $\mathcal{C}$-normalizable if the premise deductions of the last inference in $\mathcal{D}$ are all $\mathcal{C}$-normalizable. If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ is $\mathcal{C}$-normalizable if $\mathcal{D}$ has a $\mathcal{C}$-main cut and the contraction at this cut is $\mathcal{C}$-normalizable, or $\mathcal{D}$ has no $\mathcal{C}$-main cut and all minor and major deductions along the $\mathcal{C}$-main branch are $\mathcal{C}$-normalizable, and the $\mathcal{C}$-main deduction, if there is one, is also $\mathcal{C}$-normalizable.

6.4 Properties of $\mathcal{C}$-normal and $\mathcal{C}$-normalizable deductions

6.4.1 Let $\mathcal{D}$ be a $\mathcal{C}$-normal deduction on E-form. The $\mathcal{C}$-main branch of $\mathcal{D}$ will have the following properties:

(i) The minor deductions along the $\mathcal{C}$-main branch are $\mathcal{C}$-normal.

(ii) The major deductions along the $\mathcal{C}$-main branch and the $\mathcal{C}$-main deduction, if there is one, are on I-form and $\mathcal{C}$-normal.

(iii) The only inferences along the $\mathcal{C}$-main branch at which assumptions may be cancelled are $\perp$ E-inferences.

(iv) The $\mathcal{C}$-main branch ends with an assumption or with the conclusion of the $\mathcal{C}$-main deduction.

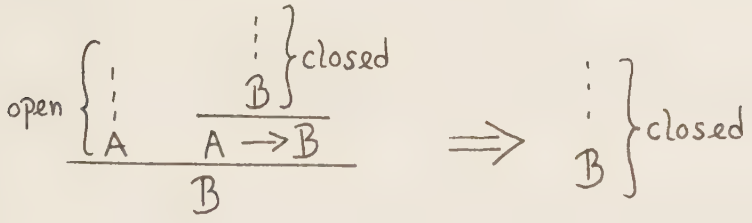
All this follows directly from the definition of " $\mathcal{D}$ is $\mathcal{C}$-normal" and " $\mathcal{C}$-main branch".

In the rest of this section we will assume that we have a fix restriction on the rules of contraction relative to predicates $\mathcal{C}$.

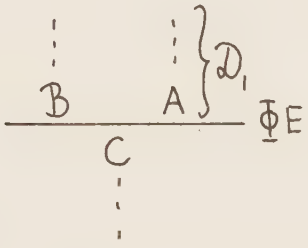
6.4.2 Let $\mathcal{C}$ be a predicate defined for deductions. I will say that $\mathcal{C}$ is reduction stable iff whenever $\mathcal{D}$ reduces to $\mathcal{D}'$ and $\mathcal{C}(\mathcal{D}')$ then $\mathcal{C}(\mathcal{D})$.

A typical example of a reduction stable predicate is the predicate " $\mathcal{D}$ reduces to a closed deduction". The predicate

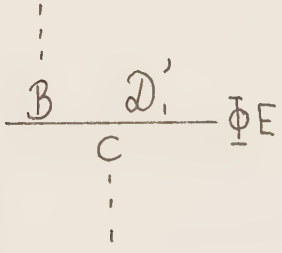
" $\mathfrak{D}$ is closed" on the other hand is not reduction stable. This is seen by the following example:



If we try to prove that $\mathfrak{C}$-normalizable deductions reduce to $\mathfrak{C}$-normal deductions by induction on the definition of " $\mathfrak{D}$ is $\mathfrak{C}$-normalizable" we will be in trouble if $\mathfrak{C}$ is not reduction stable. We may find ourselves in the situation that we know



has no $\mathfrak{C}$-main cut and so that $\mathfrak{D}_1$ will be a major deduction along the $\mathfrak{C}$-main branch. We know also that $\mathfrak{D}_1$ is $\mathfrak{C}$-normalizable and by IH (Induction Hypothesis) it then follows that $\mathfrak{D}_1$ reduces to a $\mathfrak{C}$-normal deduction $\mathfrak{D}_1'$; but then we might have a contraction defined at



if $\mathcal{C}$ is not reduction stable. And about the contraction at this cut we know nothing from the assumption that $\mathcal{D}$ is $\mathcal{C}$ -normalizable.

6.4.3 If $\mathcal{C}$ is a reduction stable predicate, then every $\mathcal{C}$ -normalizable deduction $\mathcal{D}(\Gamma;A)$ reduces to a $\mathcal{C}$ -normal deduction $\mathcal{D}'(\Gamma;A)$.

We prove this by induction on the definition of " $\mathcal{D}$ is $\mathcal{C}$ -normalizable".
If $\mathcal{D}$ is just an assumption, then $\mathcal{D}$ is $\mathcal{C}$ -normal.
If $\mathcal{D}$ is on I-form, it follows directly from IH that $\mathcal{D}$ reduces to a $\mathcal{C}$ -normal deduction
So assume $\mathcal{D}$ is on E-form:
If $\mathcal{D}$ has a $\mathcal{C}$ -main cut we know by IH that the contraction reduces to a $\mathcal{C}$ -normal deduction, and then so does $\mathcal{D}$.
If $\mathcal{D}$ has no $\mathcal{C}$ -main cut we know that all minor and major deductions along the $\mathcal{C}$ -main branch are $\mathcal{C}$ -normalizable. By IH it follows that they reduce to $\mathcal{C}$ -normal deductions.

Reducing a minor deduction to a $\mathcal{C}$ -normal deduction cannot create any new $\mathcal{C}$ -main cuts, since in this case the major premise cannot be a cut. And if $\mathcal{C}$ is reduction stable, no new $\mathcal{C}$ -main cut can be created by reducing a major deduction to a $\mathcal{C}$ -normal deduction. If there is no contraction defined at the conclusion of a major deduction along the $\mathcal{C}$ -main branch, this will not change by a reduction of the major deduction in question.
So we have a deduction with no $\mathcal{C}$ -main cut and $\mathcal{C}$ -normal minor and major deductions along the $\mathcal{C}$ -main branch.
If there is a $\mathcal{C}$ -main deduction in $\mathcal{D}$ it follows by reasoning as above that this deduction will be reducable to a $\mathcal{C}$ -normal deduction without creating any new $\mathcal{C}$ -main cut.
So $\mathcal{D}$ reduces to a $\mathcal{C}$ -normal deduction $\mathcal{D}'$.

6.5 ($\mathcal{C}$) -normalizability

In view of what has been discussed in 6.4.2 and 6.4.3 there is a need for a more general concept of normalizability relative to

a predicate $\mathfrak{C}$ if we want to choose a non reduction stable predicate $\mathfrak{C}$ for our analysis.
Such a concept will be given here. It is a weaker concept than $\mathfrak{C}$ -normalizability, so we name it $(\mathfrak{C})$-normalizability:

6.5.1 $(\mathfrak{C})$-main branch and $(\mathfrak{C})$-main cut

Let $\mathfrak{D}$ be a deduction on E-form.
I will describe the $(\mathfrak{C})$-main branch of $\mathfrak{D}$ by giving instructions how to go to the next formula from a given one, starting with the conclusion of $\mathfrak{D}$:

(i) If the given formula A is the conclusion of an E-inference and not a cut, then we proceed one step upwards to the major premise of the E-inference.

(ii) If A is a cut for which there is a rule of contraction defined, then we stop and this cut will be called the $(\mathfrak{C})$-main cut

(iii) If A is a cut for which there is no rule of contraction defined, then we stop.

(iv) If A is an assumption, then we stop.

6.5.2 $(\mathfrak{C})$-normal deductions

If $\mathfrak{D}$ consists only of an assumption, then $\mathfrak{D}$ is $(\mathfrak{C})$-normal.
If $\mathfrak{D}$ is on I-form, then $\mathfrak{D}$ is $(\mathfrak{C})$-normal if the premise deductions of the last inference in $\mathfrak{D}$ are $(\mathfrak{C})$-normal.
If $\mathfrak{D}$ is on E-form, then $\mathfrak{D}$ is $(\mathfrak{C})$-normal if $\mathfrak{D}$ has no $(\mathfrak{C})$-main cut.

6.5.3 $(\mathfrak{C})$-normalizable deductions

If $\mathfrak{D}$ consists only of an assumption, then $\mathfrak{D}$ is $(\mathfrak{C})$-normalizable.
If $\mathfrak{D}$ is on I-form, then $\mathfrak{D}$ is $(\mathfrak{C})$-normalizable if the premise deductions of the last inference in $\mathfrak{D}$ are $(\mathfrak{C})$-normalizable.

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ is $(\mathcal{C})$-normalizable if $\mathcal{D}$ has a $(\mathcal{C})$-main cut and the contraction at this cut is $(\mathcal{C})$-normalizable, or $\mathcal{D}$ has no $(\mathcal{C})$-main cut.

6.5.4 If $\mathcal{D}(\Gamma;A)$ is $(\mathcal{C})$-normalizable, then $\mathcal{D}$ reduces to a $(\mathcal{C})$-normal deduction $\mathcal{D}'(\Gamma;A)$.
This is easily seen by induction on the definition of " $\mathcal{D}$ is $(\mathcal{C})$-normalizable". The case that caused trouble in 6.4.2 was when $\mathcal{D}$ was $\mathcal{C}$-normalizable by having no $\mathcal{C}$-main cut. But if $\mathcal{D}$ has no $(\mathcal{C})$-main cut, $\mathcal{D}$ is already $(\mathcal{C})$-normal, so the problems with deductions along the $\mathcal{C}$-main branch disappear here.

6.5.5 Let $\mathcal{C}_1$ be a predicate defined for deductions such that $\mathcal{C}_1(\mathcal{D})$ implies $\mathcal{C}(\mathcal{D})$. It follows then that:
If $\mathcal{D}$ is $\mathcal{C}$-normalizable, then $\mathcal{D}$ is $(\mathcal{C}_1)$-normalizable.
This will be shown by induction on the definition of " $\mathcal{D}$ is $\mathcal{C}$-normalizable":

The cases when $\mathcal{D}$ is an assumption or on I-form follow directly, so let us concentrate on the case when $\mathcal{D}$ is on E-form:
If $\mathcal{D}$ has a $\mathcal{C}$-main cut we have two possibilities:

(i) The cut is also a $(\mathcal{C}_1)$-main cut.
Then since the contraction is $\mathcal{C}$-normalizable it follows by IH that the contraction also is $(\mathcal{C}_1)$-normalizable, so $\mathcal{D}$ is $(\mathcal{C}_1)$-normalizable.

(ii) The cut is not a $(\mathcal{C}_1)$-main cut.

Now in the $(\mathcal{C}_1)$-main branch we stop at all cuts where there is no contraction defined. So if $\mathcal{D}$ has a $(\mathcal{C}_1)$-main cut this cut has to be below the given cut, but clearly any $(\mathcal{C}_1)$-main cut will be a $\mathcal{C}$-main cut; so we may conclude that $\mathcal{D}$ has no $(\mathcal{C}_1)$-main cut and thus $\mathcal{D}$ is $(\mathcal{C}_1)$-normalizable.

If $\mathcal{D}$ has no $\mathcal{C}$-main cut we see by the previous reasoning that $\mathcal{D}$ cannot have any $(\mathcal{C}_1)$-main cut and so $\mathcal{D}$ is $(\mathcal{C}_1)$-normalizable.

6.5.6 The concepts of $\mathcal{C}$-normalizability and $(\mathcal{C})$-normalizability are designed with the intention that by the right choice of $\mathcal{C}$ we should avoid the counter examples to normalizability. If we also want to save the property that closed normal deductions are on I-form, Russell's paradox shows that we have to restrict the study to certain fragments of naive set theory.

7 Concepts of normalizability and $\mathcal{C}$-normalizability for naive set theory with intuitionistic logic

The only change needed in the treatment given in 4 - 6 consists in some additions that have to be made to the definition of $\mathcal{C}$-main branch, but since these additions will never be used in the following they are omitted here. And of course the problems in 4 - 6 connected with the classical $\perp$ E-rule disappear in the intuitionistic case,

II WELLFOUNDED FRAGMENTS OF NAIVE SET THEORY

In chapter I we saw that the formal rules of naive set theory satisfy certain rather strong inversion and reduction properties which enable us to formulate simple rules of contraction. We also saw that there are simple counter examples to normalizability for these rules of contraction for any fragment of naive set theory covering basic parts of modern set theoretical practice.
In this chapter I will discuss some fragments of naive set theory and for these fragments, given a certain predicate $\mathcal{C}$, I will give a semantical motivation for the rules of contraction connected with $\mathcal{C}$-normalizability in addition to the purely syntactical one given in chapter I.

1 Basic definitions, properties of $\mathcal{C}$-normal deduction

1.1 A fragment of naive set theory consists of a collection of terms and formulas of naive set theory together with the deductive apparatus of naive set theory.
Such a fragment N is wellfounded iff N satisfies the following conditions:

1.1.1 N contains all variables x, y, z... as terms.
$\bot$ is a formula in N.
$r \in t$, $r = t$ are formulas in N iff r and t are terms in N.
$A \& B$, $A \to B$ are formulas in N iff A and B are formulas in N.
$\forall x A(x)$ is a formula in N iff $A(x)$ is a formula in N.
If $\{x \mid A(x)\}$ is a term in N, then $A(x)$ is a formula in N.

1.1.2 If $T(x)$ is a term (formula) in N and t is a term in N, then $T(t)$ is a term (formula) in N.

1.1.3 E_N is a wellfounded relation between terms of N where E_N is defined by:

$E_N(r,t)$ iff there is a closed deduction $\mathcal{D}(r \in t)$ in N.

(As usual we will in what follows also write $\mathcal{D}(A)$ as $\vdash_N A$ and say that we have a proof of A in N.)

1.2 Now if N is a wellfounded fragment of naive set theory this system will be analyzed by the same rules of contraction as those given for naive set theory in chapter I, with the difference that these rules are not defined for all cuts for which they are defined in naive set theory itself.

In what follows we specialize $\mathcal{C}$ to be the following predicate:

$$\mathcal{C}(\mathcal{D}(T;A)) \quad \text{iff} \quad \vdash_N A .$$

The main reasons for this particular choice of a predicate $\mathcal{C}$ are:

(i) We want a closed $\mathcal{C}$-normal deduction to be on I-form, so contractions should at least be defined for cuts where the deduction of the cut and the minor deduction, if any, at the cut are closed.

(ii) The analysis we intend to give for wellfounded fragments of naive set theory will to a great extent be based on the wellfoundedness of E_N, so it seems then natural to choose $\mathcal{C}$ to be based on the global property provability rather than on a local property of deductions like " $\mathcal{D}$ is closed".

(iii) In view of 6.4.2 - 6.4.3 of chapter I it is necessary to choose a predicate $\mathcal{C}$ which is reduction stable.

(iv) In choosing provability as a basis for $\mathcal{C}$ we will get the most complete analysis related to E_N, thus covering more elementary candidates for $\mathcal{C}$ in the sense of 6.5.5 of chapter I.

(v) There are also some purely semantical motivations behind this particular choice of $\mathcal{C}$. These will be given in section 4 of this chapter.

By making this choice of $\mathcal{C}$ that we have done we have in the following to think of the concepts " $\mathcal{D}$ is $\mathcal{C}$-normal" and " $\mathcal{D}$ is $\mathcal{C}$-normalizable" as defined relative to a given fragment N.

The following restrictions on rules of contraction will be imposed for the analysis of N:

1.2.1 $\in$-contractions are defined only when the deduction of the cut satisfies $\mathcal{C}$.

1.2.2 $=$-contractions are defined as usual except when $A(x)$ is a composite $\in$-formula and x is free in $A(x)$ or $A(x) \equiv r(x) \in X$ for some r and x is free in r. In these cases contractions are defined only when both the deduction of the cut and the minor deduction at the cut satisfy $\mathcal{C}$.

1.2.3 $\perp E$-contractions are defined as usual except when the cut is on $\in$-form, then contractions are defined only when the deduction of the cut satisfies $\mathcal{C}$.

1.3 Properties of $\mathcal{C}$-normal deductions

1.3.1 Let N be a wellfounded fragment of naive set theory. If $\mathcal{D}$ is a closed $\mathcal{C}$-normal deduction in N, then $\mathcal{D}$ is on I-form.

Assume $\mathcal{D}$ is on E-form.
Following the $\mathcal{C}$-main branch in $\mathcal{D}$ upwards we stop at the first conclusion A of an inference at which some assumptions in $\mathcal{D}$ are cancelled, if there are any such formulas in the $\mathcal{C}$-main branch. We then have the following possibilities:

(i) A is the conclusion of an application of the $\perp E$-rule:

$$\dfrac{\begin{array}{c}\not{\neg A} \\ \vdots \\ \perp\end{array}}{A}$$

By the restrictions on the rules of contraction we know that this subdeduction is open if A is composite. By the wellfoundedness of E_N there can be no closed deductions ending with $r \in x$ or $\bot$, since this would mean that there is a proof of $r \in r$ contradicting the wellfoundedness of E_N. So if A is atomic, we also know that the subdeduction of $\mathcal{D}$ in question is open.

(ii) A is the minor premise of an $=E$ inference:

$$\left.\begin{array}{c} \vdots \quad\quad \vdots \\ \dfrac{A \qquad s=u}{A'} \\ \vdots \end{array}\right\} \mathcal{D}'$$

If A is atomic we use the argument of (i) to see that $\mathcal{D}'$ must be open. If A is composite on the other hand, the reason that there is no contraction defined can only be that A or $s=u$ is not provable, so $\mathcal{D}'$ has to be open.
Now if there is no such formula A in $\mathcal{D}$ as discussed above we must end up with either the $\mathcal{C}$-main deduction or an open assumption. Since $\mathcal{D}$ has no $\mathcal{C}$-main cut, the $\mathcal{C}$-main branch will end at either the $\mathcal{C}$-main deduction or at an assumption. We have assumed that there are no inferences along the $\mathcal{C}$-main branch where assumptions are cancelled, so if we end up with an assumption this must be open in $\mathcal{D}$. In case we end up with the $\mathcal{C}$-main deduction we have to consider two cases:

(i) the conclusion of the $\mathcal{C}$-main deduction is a cut. Then, because of the restrictions on the rules of contraction, this deduction must be open.

(ii) the conclusion of the $\mathcal{C}$-main deduction is the minor premise of a $=$E-inference:

$$\left.\begin{array}{c} \vdots \quad\quad \vdots \\ \dfrac{A(s) \qquad s=u}{A(u)} \\ \vdots \end{array}\right\} \mathcal{D}''$$

We know A(s) is the conclusion of an I-inference and is thus composite. The reason why no contraction is defined here can then only by that A(s) or s=u is not provable, so $\mathcal{D}''$ has to be open.

So all this means $\mathcal{D}$ must be open.

The proof of 1.3.1 uses the wellfoundedness of E_N. This is not really necessary. One could instead assume that N satisfies a certain very natural closure condition:

Let N be a fragment of naive set theory satisfying conditions 1.1.1 and 1.1.2 of this chapter.
Assume that given an atomic formula A and a proof $\mathcal{D}$ of A in N, then there is a proof $\mathcal{D}'$ of $\perp$ in N.
We have then the following:

1.3.2 Closed $\mathcal{C}$-normal deductions in N are on I-form.

Assume $\mathcal{D}$ is a closed $\mathcal{C}$-normal deduction.
Now if $\mathcal{D}$ is on E-form we follow the $\mathcal{C}$-main branch upwards in $\mathcal{D}$ as in 1.3.1.
If we come to a subdeduction on $\perp E$-form with atomic conclusion and this subdeduction is closed, then N will be inconsistent. So all formulas in N are provable and the concept of " $\mathcal{D}$ is $\mathcal{C}$-normal" collapses into the concept of " $\mathcal{D}$ is normal", and reasoning as in 4.4.2 of chapter I we see that $\mathcal{D}$ must be open. If the subdeduction on the other hand is open, we proceed upwards in $\mathcal{D}$, reasoning as in 1.3.1 until we perhaps find a new subdeduction on $\perp E$-form with atomic conclusion ... etc...

1.3.3 Let N be a wellfounded fragment of naive set theory and $\mathcal{C}_1$ a predicate defined for deductions such that if $\mathcal{D}$ is a closed deduction, then $\mathcal{C}_1(\mathcal{D})$. It then follows that:

Closed $(\mathcal{C}_1)$-normal deductions are on I-form.

This is easily seen by taking suitable parts of the argument given in 1.3.1, keeping in mind that the $(\mathcal{C}_1)$-main branch of a deduction $\mathcal{D}$ on E-form is included in the $\mathcal{C}$-main branch of $\mathcal{D}$, except at places where the $(\mathcal{C}_1)$-main branch stops, but there we have an open subdeduction of $\mathcal{D}$.

1.3.4 By our choice of $\mathcal{C}$, the concept of being a $\mathcal{C}$-normal deduction will not be decidable for most fragments N of naive set theory. But in view of 6.5.5 of chapter I and 1.3.3 above we know that an analysis of N based on $\mathcal{C}$ will in a rather strong sense cover more elementary predicates $\mathcal{C}'$ like " $\mathcal{D}$ is closed" and " $\mathcal{D}$ reduces to a closed deduction".

1.4 If we delete everything that has to do with the classical $\bot E$-rule and add the obvious clauses for v and $\exists$, the definitions of 1.1 - 1.2 run the same for fragments of naive set theory with intuitionistic logic.
In section 1.3 we observed that the use of the wellfoundedness of E_N only has to do with proper applications of the classical $\bot E$-rule. So in the intuitionistic case we see that there are no assumptions cancelled along the $\mathcal{C}$-main branch, and using parts of the argument given in 1.3.1 we thus have an elementary argument establishing that closed $\mathcal{C}$-normal deductions are on I-form. 1.3.2 - 1.3.4 are of course still true in this case.

1.5 Notes

1.3.2 is due to Torkel Franzén.

2 Axioms for set theory

I will now set out to give examples of wellfounded fragments of naive set theory based on first order axiom systems for set theory and the assumption that there are models of these systems in which the interpretation of $\in$ is wellfounded.

2.1 A formal language L

The axiom systems for set theory considered here will be formulated in a subsystem of naive set theory given by the following clauses for forming terms and formulas:

Variables x, y, z... are terms.
$\bot$ is a formula.
$x =_L y$ is a formula if x and y are variables.
If t is a term, x is a variable, then $x \in_L t$ is a formula.
If A and B are formulas, then A & B and $A \to B$ are formulas.
If $A(x)$ is a formula, then $\forall x A(x)$ is a formula and $\{x|A(x)\}$ is a term.
If r or t is an abstraction term, then
$r =_L t \underset{def.}{=} \forall x (x \in_L r \leftrightarrow x \in_L t)$.
If r is an abstraction term, then
$r \in t \underset{def.}{=} \exists x (x =_L r \& x \in_L t)$.
Let $V \underset{def.}{=} \{x | x =_L x\}$. It is easy to see that $t \in_L V$ is equivalent to $\exists x (x =_L t)$.

2.2 Semantics for L

A model $\mathcal{A}$ for L is a first order structure $\langle D_{\mathcal{A}}, \in_{\mathcal{A}}, =_{\mathcal{A}} \rangle$ in which the axiom of extensionality is true.

Let S be a function from the collection of variables in L to the domain of $\mathcal{A}$,

$\mathcal{A} \models A(\bar{x})[s]$ (s satisfies $A(\bar{x})$ in $\mathcal{A}$)

is defined as usual for atomic formulas on $=_L$ or $\in_L$-form and for formulas on $\&$, $\rightarrow$ and $\forall$-form. To complete the definition add the following two clauses:

$\mathcal{A} \models \bot [s]$ is false

$\mathcal{A} \models x \in_L \{z \mid A(z, \bar{x})\}$ iff $\mathcal{A} \models A(x, \bar{x})[s]$.

As a formal system for L we use the formal system given for naive set theory in section 1.2 of chapter I with the following restriction on the $\forall E$-rule:

$$\frac{\vdots}{\dfrac{\forall_x A(x)}{A(z)}}$$

As usual we may add axioms to form theories Z in L.

It is easily seen by induction on the length of a deduction $\mathcal{D}$ that the formal system for L is sound relative to the above semantics.

2.3 Interpretation of fragments of naive set theory in L

2.3.1 Let T be a term (formula) of naive set theory. By induction on the construction of T I will define a term (formula) T^* in L:

If $T \equiv x$, then $T^* \underset{def.}{=} x$.
If $T \equiv \bot$, then $T^* \underset{def.}{=} \bot$.
If $T \equiv r \in t$, then $T^* \underset{def.}{=} r^* \in_L t^*$.
If $T \equiv r = t$, then $T^* \underset{def.}{=} r^* =_L t^*$.
If $T \equiv A \& B$, then $T^* \underset{def.}{=} A^* \& B^*$.
If $T \equiv A \rightarrow B$, then $T^* \underset{def.}{=} A^* \rightarrow B^*$.
If $T \equiv \forall x A(x)$, then $T^* \underset{def.}{=} \forall x A(x)^*$.
If $T \equiv \{x \mid A(x)\}$, then $T^* \underset{def.}{=} \{x \mid A(x)^*\}$.

2.3.2 If $T(x)$ is a term (formula) of naive set theory $T(x)^*[r^*/x]$ will be the following term (formula) of L for a term r in naive set theory where $\vdash_N r^* \in_L V$:

If $T(x) \equiv x$, then $T(x)^*[r^*/x] =_{def.} r^*$.

If $T(x) \equiv z$, then $T(x)^*[r^*/x] =_{def.} z$.

If $T(x) \equiv A(x)$, then $T(x)^*[r^*/x] =_{def.} \exists x(x =_L r^* \& A(x)^*)$.

If $T(x) \equiv \{z \mid A(z,x)\}$, then $T(x)^*[r^*/x] =_{def.} \{z \mid \exists y(y =_L r^* \& z \in_L \{v \mid A(v,y)^*\}\}$.

By induction on the construction of $T(x)$ I will show that if $T(x)$ is a formula, then $\vdash_L T(r)^* \leftrightarrow T(x)^*[r^*/x]$ and if $T(x)$ is a term, then $\vdash_L T(r)^* =_L T(x)^*[r^*/x]$.

2.3.2.1 $T(x) \equiv x$. Obviously $\vdash_L r^* =_L r^*$.
And if $T(x) \equiv z$, we have $\vdash_L z =_L z$.

2.3.2.2 $T(x) \equiv \perp$. trivial.

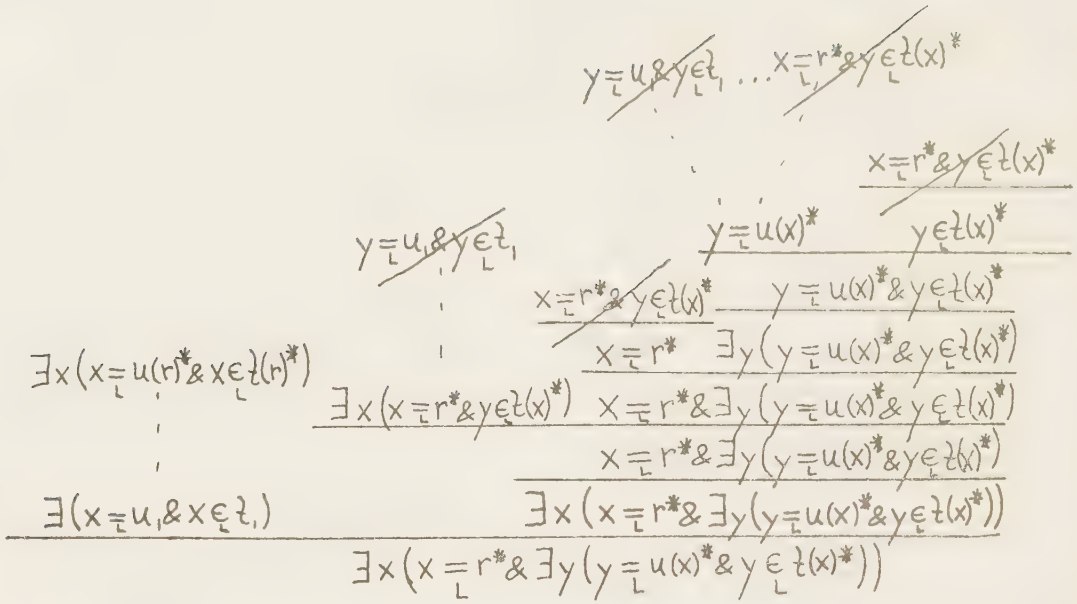
2.3.2.3 $T(x) \equiv u(x) \in t(x)$.

Let $t_1 =_{def.} \{z \mid \exists y(y =_L r^* \& z \in_L t(y)^*)\}$

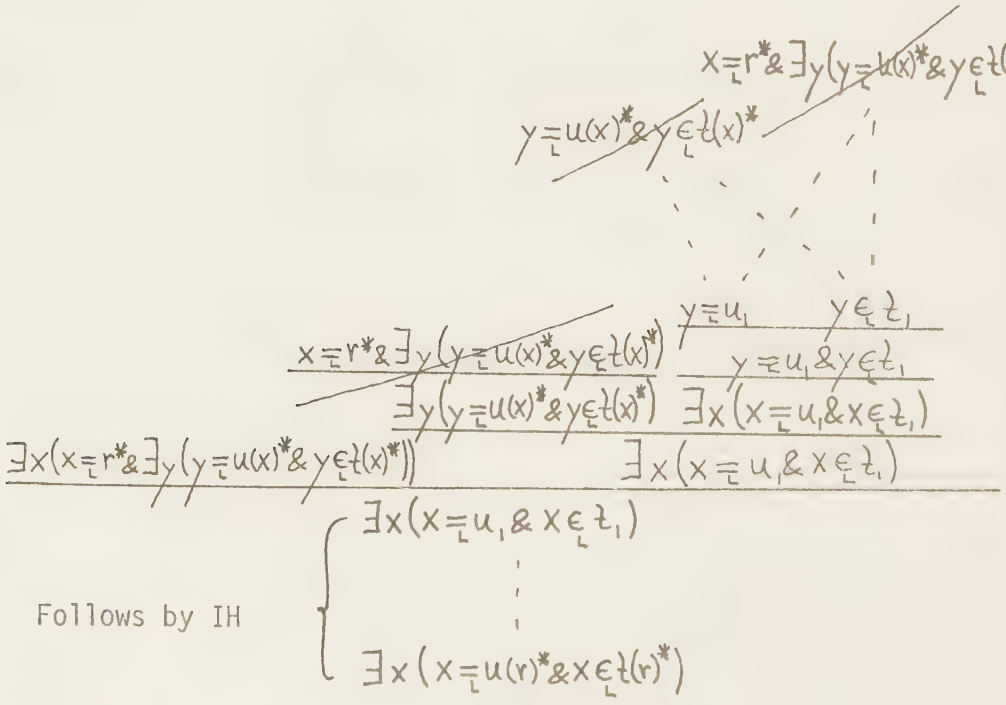
$u_1 =_{def.} \{z \mid \exists y(y =_L r^* \& z \in_L u(y)^*)\}$.

By IH $\vdash_L t_1 =_L t(r)^*$, $\vdash_L u_1 =_L u(r)^*$.

Now it is easily seen that we have a deduction of $\exists x(x =_L u_1 \& x \in_L t_1)$
from $\exists x(x =_L u(r)^* \& x \in_L t(r)^*)$
and it is not hard to see that we can deduce $y =_L u(x)^*$
from $x =_L r^* \& y \in_L t(x)^*$ and $y =_L u_1 \& y \in_L t_1$
so then we have the following deduction of
$\exists x(x =_L r^* \& \exists y(y = u(x)^* \& y \in t(x)^*))$
from $\exists x(x = u(r)^* \& x \in t(r)^*)$:



In the other direction it is easily seen that $y \subseteq u_1$ and $y \in u$ are deducible from $y \subseteq u(x)^* \,\&\, y \in t(x)^*$ and $x \subseteq r^* \,\&\, \exists y (y \subseteq u(x)^* \,\&\, y \in t(x)^*)$, so then we have the following deduction of $\exists x (x \subseteq u(r)^* \,\&\, x \in t(r)^*)$ from $\exists x (x \subseteq r^* \,\&\, \exists y (y \subseteq u(x)^* \,\&\, y \in t(x)^*))$:



2.3.2.4 Since the $\&$, $\to$ and $\forall$ -cases are treated similarly, I will only consider the $\forall$ -cases here as an example.

Now by assumption $\vdash_L r^* \in_L V$, so we have a closed duction of $\exists z\,(z =_L r^*)$.

It is easy to see that there is a deduction of $x =_L x'$ from $x =_L r^*$ and $x' =_L r^*$, so then we have the following deduction of $\exists y\,(y =_L r^* \,\&\, \forall z\, A(z,y)^*)$ from $\forall z\, A(z,r)^*$;

$$\dfrac{\dfrac{\dfrac{

$$\dfrac{\exists x(x\mathrel{=\!\!\!\subset} r^* \,\&\, \forall z\, A(z,x)^*)\quad A(z,r)^*}{\dfrac{A(z,r)^*}{\forall z\, A(z,r)^*}}$$

where the left premiss is obtained from

$$\dfrac{\dfrac{\dfrac{x\mathrel{=\!\!\!\subset} r^* \,\&\, \forall z\, A(z,x)^*}{x\mathrel{=\!\!\!\subset} r^*}\qquad \dfrac{x\mathrel{=\!\!\!\subset} r^* \,\&\, \forall z\, A(z,x)^*}{\forall z\, A(z,x)^*}}{\dfrac{x\mathrel{=\!\!\!\subset} r^* \,\&\, A(z,x)^*}{A(z,x)^*}}}{\dfrac{x\mathrel{=\!\!\!\subset} r^* \,\&\, A(z,x)^*}{\exists x(x\mathrel{=\!\!\!\subset} r^* \,\&\, A(z,x)^*)}}$$

by IH.

2.3.2.5 The $\mathrel{=\!\!\!\subset}$-case is treated as a combination of the $\to$ and $\forall$-cases, so let us finally look at the case when $T(x) \equiv \{z \mid A(z,x)\}$:

First we have to show that there is a deduction from $v \in \{z \mid A(z,r)^*\}$ to $v \in \{z \mid \exists y(y \mathrel{=\!\!\!\subset} r^* \,\&\, z \in \{x \mid A(x,y)^*\}\}$:

$$\dfrac{\exists y(y\mathrel{=\!\!\!\subset} r^* \,\&\, A(v,y)^*)\quad \exists y(y\mathrel{=\!\!\!\subset} r^* \,\&\, v\in\{x\mid A(x,y)^*\})}{\exists y(y\mathrel{=\!\!\!\subset} r^* \,\&\, v\in\{x\mid A(x,y)^*\})}$$

$$\dfrac{}{v \in \{\exists y(y\mathrel{=\!\!\!\subset} r^* \,\&\, z\in\{x\mid A(x,y)^*\})\}}$$

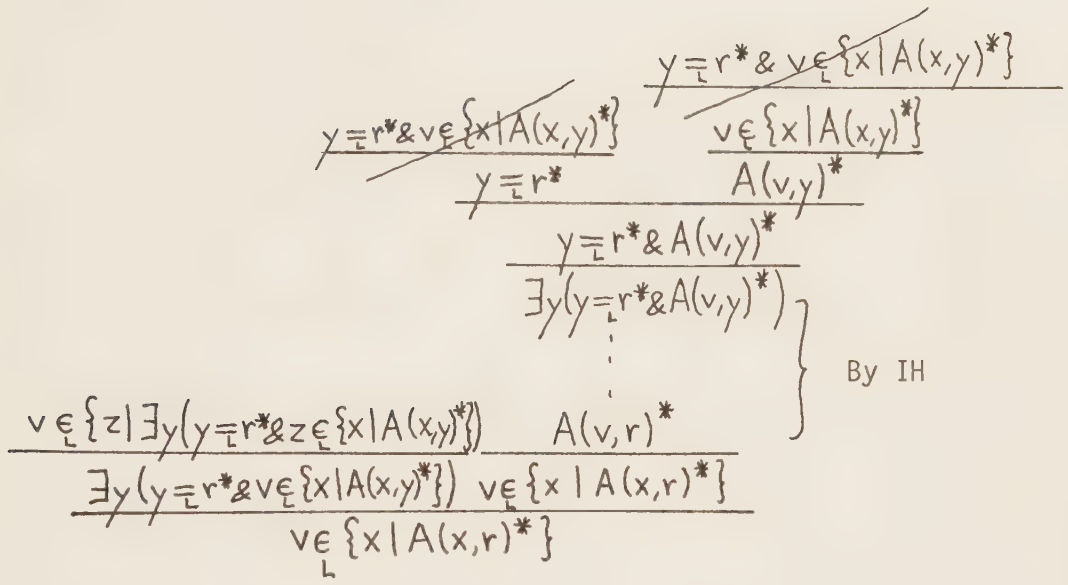
where the left premiss is obtained from

$$\dfrac{\dfrac{v\in\{z\mid A(z,r)^*\}}{A(v,r)^*}}{\exists y(y\mathrel{=\!\!\!\subset} r^* \,\&\, A(v,y)^*)}\ \text{By IH}$$

and the right premiss is obtained from

$$\dfrac{\dfrac{\dfrac{y\mathrel{=\!\!\!\subset} r^* \,\&\, A(v,y)^*}{y\mathrel{=\!\!\!\subset} r^*}\qquad \dfrac{y\mathrel{=\!\!\!\subset} r^* \,\&\, A(v,y)^*}{A(v,y)^*}}{y\mathrel{=\!\!\!\subset} r^* \,\&\, v\in\{x\mid A(x,y)^*\}}\qquad \dfrac{y\mathrel{=\!\!\!\subset} r^* \,\&\, A(v,y)^*}{v\in\{x\mid A(x,y)^*\}}}{\exists y(y\mathrel{=\!\!\!\subset} r^* \,\&\, v\in\{x\mid A(x,y)^*\})}$$

and secondly that we have a deduction in the other direction:



2.3.3 Z-fragments of naive set theory

Let Z be an axiom system for L.
If N is a fragment of naive set theory such that $\vdash_Z t^* \in_L V$ for all terms t in N, then N is said to be a Z-fragment of naive set theory.

2.3.4 If N is a Z-fragment of naive set theory, then $A_1 \ldots A_n \vdash_N A$ implies $A_1^* \ldots A_n^* \vdash_Z A^*$.

This will be proved by induction on the length of the deduction $\mathfrak{D}(A_1 \ldots A_n; A)$ in N:

2.3.4.1 If $\mathcal{D}$ consists only of an assumption, then clearly the claim is correct.

2.3.4.2 $\mathcal{D}$ is on $\in$ I-form

$$\dfrac{\begin{array}{c}\vdots\\ A(r)\end{array}}{r\in\{x\mid A(x)\}}$$

By IH there is a deduction $\begin{array}{c}\vdots\\ A(r)^*\end{array}$

Using 2.3.2 we thus obtain the following deduction of $\left(r\in_L\{x\mid A(x)\}\right)^*$:

$$\dfrac{\exists x(x=_L r^*\ \&\ A(x)^*)\qquad \exists x(x=_L r^*\ \&\ x\in_L\{x\mid A(x)^*\})}{\exists x(x=_L r^*\ \&\ x\in_L\{x\mid A(x)^*\})}$$

where the left premise is $\left\{\begin{array}{c}\vdots\\ A(r)^*\end{array}\right.$ and the right premise is

$$\dfrac{\dfrac{\not{x=_L r^*\ \&\ A(x)^*}}{x=_L r^*}\qquad \dfrac{\not{x=_L r^*\ \&\ A(x)^*}}{A(x)^*}}{x=_L r^*\ \&\ x\in_L\{x\mid A(x)^*\}}\qquad x\in_L\{x\mid A(x)^*\}$$

2.3.4.3 $\mathcal{D}$ is on $=$ I-form

$$\dfrac{\begin{array}{c}\not{x\in r}\\ \vdots\\ x\in t\end{array}\qquad\qquad \begin{array}{c}\not{x\in t}\\ \vdots\\ x\in r\end{array}}{r=t}$$

By IH we have $x \in_L r^*$ ⋮ $x \in_L t^*$ and $x \in_L t^*$ ⋮ $x \in_L r^*$

so clearly $(r =_L t)^*$ follows in L.

2.3.4.4 $\mathcal{D}$ is on & I-form

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A\&B} \qquad \text{By IH} \quad \begin{array}{cc}\vdots & \vdots \\ A^* & B^*\end{array}$$

so clearly $(A\&B)^*$ follows in L.

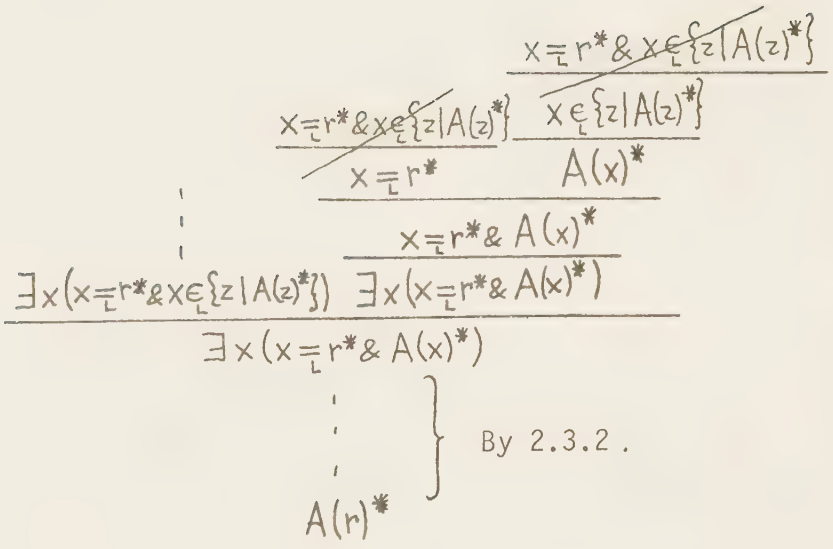
The cases when $\mathcal{D}$ is on $\rightarrow$I and $\forall$-I form follows in a similar manner directly from IH.

2.3.4.5 $\mathcal{D}$ is on $\in$ E-form

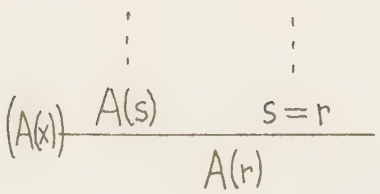
$$\frac{\begin{array}{c}\vdots \\ r \in \{x \mid A(x)\}\end{array}}{A(r)} \qquad \begin{array}{c}\vdots \\ \end{array}$$

By IH we have a deduction $r^* \in_L \{x \mid A(x)^*\}$,

so using 2.3.2 we have the following deduction of $A(r)^*$:



2.3.4.6 $\mathcal{D}$ is on $=$E-form

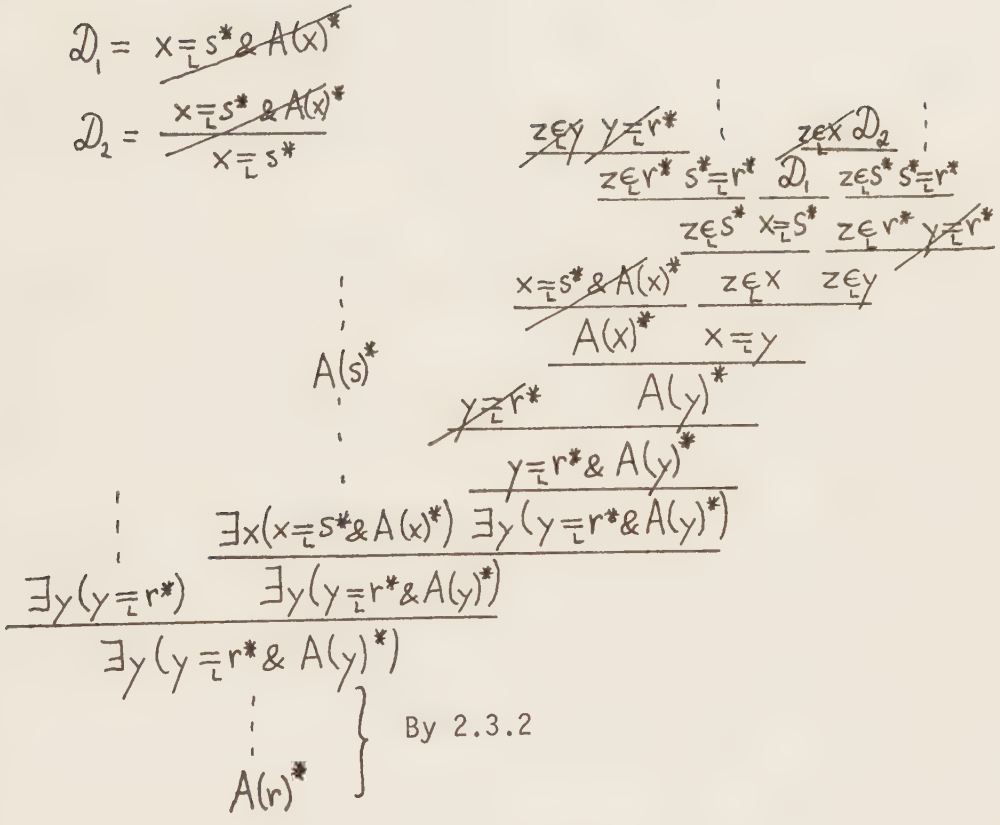


By IH $A(s)^*$, $(s=r)^*$.

Now it is easily seen that $\dfrac{z \in_L t \quad t =_L r}{z \in_L r}$

is valid as a derived rule in L.

Using this, 2.3.2 and the fact that $\vdash_L r^* \in_L V$ we obtain the following deduction of $A(r)^*$:


$\mathcal{D}_1 = x \subseteq_L s^* \;\&\; A(x)^*$
$\mathcal{D}_2 = \dfrac{x \subseteq_L s^* \;\&\; A(x)^*}{x \subseteq_L s^*}$
$z \in_L y \quad y \subseteq_L r^*$
$z \in_L r^* \quad s^* \subseteq_L r^*$
$z \in_L x \quad \mathcal{D}_2$
$z \in_L s^* \quad \mathcal{D}_1 \quad z \in_L s^* \quad s^* \subseteq_L r^*$
$z \in_L s^* \quad x \subseteq_L s^* \quad z \in_L r^* \quad y \subseteq_L r^*$
$x \subseteq_L s^* \;\&\; A(x)^* \quad z \in_L x \quad z \in_L y$
$A(x)^* \quad x = y$
$A(y)^*$
$y \subseteq_L r^* \quad A(y)^*$
$y \subseteq_L r^* \;\&\; A(y)^*$
$A(s)^*$
$\exists x(x \subseteq_L s^* \;\&\; A(x)^*) \quad \exists y(y \subseteq_L r^* \;\&\; A(y)^*)$
$\exists y(y \subseteq_L r^*) \quad \exists y(y \subseteq_L r^* \;\&\; A(y)^*)$
$\exists y(y \subseteq_L r^* \;\&\; A(y)^*)$
$A(r)^*$
By 2.3.2

2.3.4.7 $\mathcal{D}$ is on & E-form

$$\frac{\begin{array}{c}\vdots\\ A \,\&\, B\end{array}}{B} \qquad \text{By IH } A^* \,\&\, B^* \text{ so clearly}$$

B^* follows in L.

The case when $\mathcal{D}$ is on $\rightarrow$ E-form follows in a similar manner directly from IH.

2.3.4.8 $\mathcal{D}$ is on $\forall$ E-form

$$\dfrac{\begin{array}{c}\vdots\\ \forall x\, A(x)\end{array}}{A(t)}$$

By IH $\quad \begin{array}{c}\vdots\\ \forall x A(x)^*\end{array}$ Now using 2.3.2 and the fact that $\vdash_Z t^* \in_L V$ we obtain the following deduction of $A(t)^*$:

$$\dfrac{\dfrac{\dfrac{\exists y(y =_L t^*) \quad \exists y(y =_L t^* \& A(y)^*)}{\exists y\,(y =_L t^* \& A(y)^*)}}{\begin{array}{c}\vdots\\ A(t)^*\end{array}}\ \text{By 2.3.2.}}$$

where the left premiss $\exists y(y =_L t^*)$ is obtained from $\begin{array}{c}\vdots\\ t^* \in V\end{array}$, and the right premiss $\exists y(y =_L t^* \& A(y)^*)$ is obtained from

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall x\, A(x)^*\end{array}}{A(y)^*} \qquad \not{y =_L t^*}}{y =_L t^* \& A(y)^*}$$

2.3.4.9 $\mathcal{D}$ is on $\bot$ E-form

$$\dfrac{\dfrac{\not{\neg A}}{\begin{array}{c}\vdots\\ \bot\end{array}}}{A}$$

By IH $\quad \begin{array}{c}\neg A^*\\ \vdots\\ \bot\end{array}$, so clearly A^* follows in L cancelling $\neg A^*$.

2.4 Wellfounded Z-fragments of naive set theory

Let N be a Z-fragment such that conditions 1.1.1 and 1.1.2 of this chapter are satisfied. Now if there is a model of Z in which $\in_L$ is interpreted as a wellfounded relation, then N is a wellfounded fragment of naive set theory. That is E_N is wellfounded: To see this, assume there is an infinite chain $t_1, t_2 \ldots$ of terms in N such that $\vdash_N t_{i+1} \in_L t_i$ for all i.

By 2.3.4 $\vdash_Z t_{i+1}^* \in_L t_i^*$. Since $\vdash_Z t_i^* \in_L V$ for all i there will be an infinite chain of elements $a_1, a_2 \ldots$ such that $a_{i+1} \in_L^{\mathcal{A}} a_i$ for any model $\mathcal{A}$ of Z, contradicting the assumption that there is a model of Z in which $\in_L$ is given a wellfounded interpretation.

2.5 Zermelo-Fraenkel set theory

2.5.1 Let ZF be the following axioms and axiom schemata for L:

A 1 $\{x \mid x \neq x\} \in_L V$

A 2 $\forall x \forall y ((x \in_L V \& y \in_L V) \rightarrow \{z \mid z =_L x \vee z =_L y\} \in_L V)$.

A 3 $\forall x (x \in_L V \rightarrow \{y \mid \exists z (z \in_L x \& y \in_L z)\} \in_L V)$.

A 4 $\forall x (x \in_L V \rightarrow \{y \mid \forall z (z \in_L y \rightarrow z \in_L x)\} \in_L V)$.

A 5 $\forall x (x \in_L V \rightarrow \{y \mid y \in x \& A(y)\} \in_L V)$. x not free in $A(y)$.

A 6 $\forall x (x \in_L V \rightarrow \{y \mid \exists z ((z \in_L x \& A(z,y)) \& \forall u (A(z,u) \rightarrow y =_L u))\} \in_L V)$ x not free in $A(z,y)$.

A 7 $\exists x (x \in_L V \& (\{x_1 \mid x_1 \neq_L x_1\} \in_L x \& \forall y (y \in_L x \rightarrow \rightarrow \{z \mid \exists u (u \in_L \{x_2 \mid x_2 =_L y \vee x_2 =_L \{x_3 \mid x_3 =_L y\}\} \& z \in_L u)\} \in_L x)))$.

In view of A 1 - A 7 is easy to see that the following holds for ZF:

2.5.1.1 If $\vdash_{ZF} r \in_L V$ and $\vdash_{ZF} t \in_L V$, then $\vdash_{ZF} \{x \mid x =_L r \vee x =_L t\} \in_L V$.

2.5.1.2 If $\vdash_{ZF} t \in_L V$, then $\vdash_{ZF} \{x \mid \exists y (y \in_L t \,\&\, x \in_L y\} \in_L V$.

2.5.1.3 If $\vdash_{ZF} t \in_L V$, then $\vdash_{ZF} \{x \mid \forall_y (y \in_L x \to y \in_L t)\} \in_L V$.

2.5.1.4 If $\vdash_{ZF} t \in_L V$, then $\vdash_{ZF} \{x \mid x \in_L t \,\&\, A(x)\} \in_L V$.

Since all the cases 2.5.2.1 - 2.5.2.5 are treated in a uniform manner we can take 2.5.1.4 as an example:

Consider the following two deductions

$$\mathcal{D}_1 \left\{ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v \in_L \{y \mid y \in_L t \,\&\, A(y)\}}{v \in_L t \,\&\, A(v)}}{v \in_L t} \quad x =_L t}{v \in_L x} \quad \dfrac{\mathcal{D}}{A(v)}}{v \in_L x \,\&\, A(v)}}{v \in_L \{y \mid y \in_L x \,\&\, A(y)\}} \quad z =_L \{y \mid y \in_L x \,\&\, A(y)\}}{v \in_L z} \right.$$

$$\mathcal{D}_2\left\{\begin{array}{l}\dfrac{\dfrac{\dfrac{\dfrac{v\in z \qquad z=\{y\mid y\in x\,\&\,A(y)\}}{v\in\{y\mid y\in x\,\&\,A(y)\}}}{v\in x\,\&\,A(v)}\;\mathcal{D}'}{v\in x \qquad x=t}\qquad \dfrac{\mathcal{D}'}{A(v)}}{v\in t\,\&\,A(v)}\\[2ex] \hline v\in\{y\mid y\in t\,\&\,A(y)\}\end{array}\right.$$

Then we have the following deduction of $\exists z(z=\{y\mid y\in t\,\&\,A(y)\})$:

$$\dfrac{\exists x(x=t) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\forall x(x\in V\to\exists z(z=\{y\mid y\in x\,\&\,A(y)\}))} \qquad \dfrac{\mathcal{D}_1}{x\in V \quad x\in V\to\exists z(z=\{y\mid y\in x\,\&\,A(y)\})}}{\exists z(z=\{y\mid y\in x\,\&\,A(y)\})} \qquad \dfrac{\mathcal{D}_2}{z=\{y\mid y\in t\,\&\,A(y)\}}}{\exists z(z=\{y\mid y\in t\,\&\,A(y)\})}}{\exists z(z=\{y\mid y\in t\,\&\,A(y)\})}$$

2.5.1.5 If $\vdash_{ZF} t\in_L V$, then

$$\vdash_{ZF}\{y\mid \exists z((z\in_L t\,\&\,A(z,y))\,\&\,\forall u(A(z,u)\to y=u))\}\in_L V.$$

2.5.1.6 As a consequence of A 7 and A 5 we have

$$\vdash_{ZF} \{x \mid \forall y\,(\{x_1 \mid x_1 \neq_L x_1\} \in_L y \,\&\, \forall z(z \in_L y \rightarrow$$
$$\rightarrow \{x_2 \mid \exists u(u \in_L \{x_3 \mid x_3 =_L z \vee x_3 =_L \{x_4 \mid x_4 =_L z\}\}\,\&$$
$$\&\, x_2 \in_L u\} \in_L y) \rightarrow x \in_L y)\} \in_L V.$$

2.5.2 I will now define a fragment N_{ZF} of naive set theory and show that N_{ZF} is a ZF-fragment.

2.5.2.1 Terms and formulas of N_{ZF} :

Variables x, y, z... are terms.
$\perp$ is a formula.
If r, t are terms, then $r=t$ and $r \in t$ are formulas.
If A, B are formulas, then A & B and $A \rightarrow B$ are formulas.
If A(x) is a formula, then $\forall x A(x)$ is a formula.
$\{x \mid x \neq x\}$ is a term $(\emptyset)$.
If r, t are terms, then $\{x \mid x=r \vee x=t\}$ $(\{r,t\})$
$\{x \mid \exists y(y \in r \,\&\, x \in y)\}$ (Ur) and
$\{x \mid \forall y(y \in x \rightarrow y \in r)\}$ (Pr) are terms.
If r is a term and A(x) a formula, then $\{x \mid x \in r \,\&\, A(x)\}$ is a term.
If r is a term and A(x, y) a formula, then
$\{x \mid \exists y((y \in r \,\&\, A(y,x)) \,\&\, \forall u(A(y,u) \rightarrow x=u))\}$
is a term $(A(,)[r])$.
$\{x \mid \forall y(\emptyset \in y \,\&\, \forall z(z \in y \rightarrow U\{z, \{z\}\} \in y) \rightarrow x \in y)\}$
is a term (ω).
x is not free in r or t.
$\exists$ and v are defined as in naive set theory.

2.5.2.2 N_{ZF} is a ZF-fragment.

We have to show $\vdash_{ZF} t^* \in_L V$ if t is a term of N_{ZF}. This will be done by induction on the construction of t:

If $t \equiv x$ the claim is trivially true.
If t is composite we have to treat several cases:

2.5.2.2.1 $t \equiv \{x \mid x \neq x\}$.

$t^* = t$, so by A1 $\vdash_{ZF} t^* \in_L V$.

2.5.2.2.2 $t \equiv \{x \mid x = r \vee x = s\}$.

By IH $\vdash_{ZF} r^* \in_L V$, $\vdash_{ZF} s^* \in_L V$. Using 2.5.3.1 we see that

$\{x \mid x =_L r^* \vee x =_L s^*\} \in_L V$ which is what we want.

2.5.2.2.3 $t \equiv \{x \mid \exists y (y \in r \& x \in y)\}$.

By IH $\vdash_{ZF} r^* \in_L V$ so using 2.5.1.2 we know

$\vdash_{ZF} \{x \mid \exists y (y \in_L r^* \& x \in_L y)\} \in_L V$ which is just $\vdash_{ZF} t^* \in_L V$.

2.5.2.2.4 $t \equiv \{x \mid \forall y (y \in x \rightarrow y \in r)\}$.

This case is treated like 2.5.2.2.3.

2.5.2.2.5 $t \equiv \{x \mid x \in r \& A(x)\}$.

By IH $\vdash_{ZF} r^* \in_L V$. Now $t^* = \{x \mid x \in_L r^* \& A(x)^*\}$,

so using 2.5.1.4 we see that $\vdash_{ZF} t^* \in_L V$.

2.5.2.2.6 $t \equiv \{x \mid \exists y ((y \in r \& A(y,x)) \& \forall z (A(y,z) \rightarrow z = x))\}$.

This case is treated like 2.5.2.2.5

2.5.2.2.7 $t \equiv \{x \mid \forall y (\emptyset \in y \& \forall z (z \in y \rightarrow \cup\{z, \{z\}\} \in y) \rightarrow x \in y)\}$.

That $\vdash_{ZF} t^* \in_L V$ follows from 2.5.1.6.

2.5.2.3 So by 2.5.2.2 if ZF has a model in which $\in$ is given a wellfounded interpretation, then N_{ZF} is a wellfounded fragment of naive set theory, provided N_{ZF} satisfies 1.1.1 and 1.1.2 of this chapter.

Clearly 1.1.1 is satisfied, so to see that 1.1.2 is also satisfied, assume $T(x)$ is a term (formula) of N_{ZF} and t a term of N_{ZF}. We have then to make sure that $T(t)$ is a term (formula) of N_{ZF}. That this really is so is easily seen by induction on the construction of $T(x)$.

2.5.2.4 By inspection of the forms of the terms of N_{ZF} we can easily see that the following rules of inference are valid as derived rules in N_{ZF}:

2.5.2.4.1 $\emptyset$ -elimination

$$\frac{\begin{array}{c}\vdots\\ r\in\emptyset\end{array}}{\bot}$$

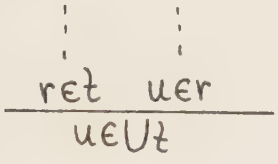
2.5.2.4.2 $\{\ ,\ \}$ -introduction

$$\frac{\begin{array}{c}\vdots\\ r=u\end{array}}{r\in\{t,u\}}$$

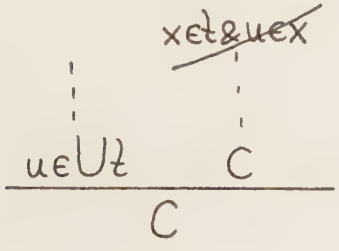
2.5.2.4.3 $\{\ ,\ \}$ -elimination

$$\frac{\begin{array}{c}\vdots\\ r\in\{t,u\}\end{array}\qquad \begin{array}{c}\not{r=t}\\ \vdots\\ C\end{array}\qquad \begin{array}{c}\not{r=u}\\ \vdots\\ C\end{array}}{C}$$

2.5.2.4.4 $\bigcup$-introduction

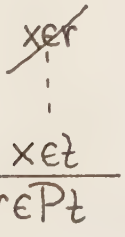


2.5.2.4.5 $\bigcup$-elimination



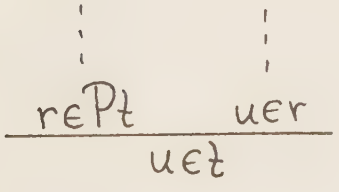
restrictions on x as in $\exists$ E-inferences.

2.5.2.4.6 $\mathcal{P}$-introduction



restrictions on x as in $\forall$ I-inferences with exceptions for assumptions $x \in r$.

2.5.2.4.7 $\mathcal{P}$-elimination



2.5.2.4.8 $\{x \mid x\epsilon t \& A(x)\}$ -introduction

$$\frac{\begin{array}{cc} \vdots & \vdots \\ r\epsilon t & A(r) \end{array}}{r\epsilon\{x \mid x\epsilon t \& A(x)\}}$$

2.5.2.4.9 $\{x \mid x\epsilon t \& A(x)\}$ -elimination

$$\frac{\vdots}{\dfrac{r\epsilon\{x \mid x\epsilon t \& A(x)\}}{A(r)}} \qquad \frac{\vdots}{\dfrac{r\epsilon\{x \mid x\epsilon t \& A(x)\}}{r\epsilon t}}$$

2.5.2.4.10 $A(\ ,\ )[\ ]$ -introduction

$$\frac{\begin{array}{ccc} \cancel{A(x,y)\&A(x,z)} & \vdots & \vdots \\ \vdots & & \\ y=z & A(t,r) & t\epsilon u \end{array}}{r\epsilon A(\ ,\ )[u]}$$

2.5.2.4.11 $A(\ ,\ )[\ ]$ -elimination

restriction on x
as in $\exists$ E-inferences.

$$\frac{\begin{array}{cc} \vdots & \cancel{x\epsilon u \& A(x,r)} \\ & \vdots \\ r\epsilon A(\ ,\ )[u] & C \end{array}}{C}$$

2.5.2.4.12 ω -introduction

restrictions on x as in $\forall$ I with exceptions for discharged assumptions

$$\dfrac{\begin{array}{c}\vdots\\ \emptyset\in x \;\&\; \forall y\,(y\in x \rightarrow \cup\{y,\{y\}\}\in x)\end{array}}{r\in x}$$

$$r\in\omega$$

2.5.2.4.13 ω -elimination

$$\dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ r\in\omega & \emptyset\in t & \forall x(x\in t \rightarrow \cup\{x,\{x\}\}\in t)\end{array}}{r\in t}$$

So each of the axioms and axiom schemata A 1 - A 7 is represented in N_{ZF} by a rule or pair of rules of inference, this being perhaps the canonical way of representing ZF as a system of natural deduction.

2.5.3 To compare N_{ZF} with ZF, we know by 2.5.2.2 and 2.3.4 that $A_1 \ldots A_n \vdash_{N_{ZF}} A$ implies $A_1^* \ldots A_n^* \vdash_{ZF} A^*$.

On the other hand it is easily seen by induction on the construction of $\ulcorner T \urcorner$ that if $\ulcorner T \urcorner$ is a formula, then $\vdash_{N_{ZF}} \ulcorner T \urcorner \leftrightarrow \ulcorner T \urcorner^*$ and if $\ulcorner T \urcorner$ is a term, then $\vdash_{N_{ZF}} \ulcorner T \urcorner = \ulcorner T \urcorner^*$. Now by the very construction of N_{ZF} it follows that the axioms of ZF are provable

in N_{ZF} when we substitute $\exists(x=r)$ for $r \in \overset{L}{V}$ to make the axioms formulas in N_{ZF}. So for formulas $A_1 \ldots A_n, A$ in N_{ZF} we have that $A_1^* \ldots A_n^* \vdash_{ZF} A^*$ implies $A_1 \ldots A_n \vdash_{N_{ZF}} A$.

2.6 Notes

The idea of formulating Fraenkel's replacement axiom in terms of an abstraction term is taken from Powell([P]).

3 The language SET

In this section I will give an example of a wellfounded fragment of naive set theory which is a kind of higher order language constructed directly over the cumulative hierarchy. The basic idea is simply to take all the definable sets in the cumulative hierarchy as terms in the sense that if all the sets satisfying $A(x)$ form a set, then $\{x \mid A(x)\}$ will be a term. Of course such a language will be a bit more abstract than for example N_{ZF}, but still I think it gives a good picture of what is needed in order to construct a language of set theory and to give it a semantic.

3.1 The structure C

To build up C I will assume the collection of ordinals ON contains $\emptyset$, is closed under successor operation and also contains ω.

$$C_\emptyset \underset{def.}{=} \emptyset$$

$$C_{\alpha+1} \underset{def.}{=} C_\alpha \cup \mathcal{P}(C_\alpha)$$

$$C_\lambda \underset{def.}{=} \bigcup_{\beta<\lambda} C_\beta \qquad \lambda \text{ limit ordinal}$$

$$C \underset{def.}{=} \bigcup_{\alpha \in ON} C_\alpha .$$

I will assume there are relations $\underset{C}{=}, \in_C$ on C with their canonical meaning of extensional equility and membership.

3.2 The language SET over C

Given variables x, y, z... and functions ϕ from the class of variables to C, I will define simultaneously $T(\bar{x})$ is a term (formula) of SET, $C \models T(\bar{x})[\phi]$ (ϕ satisfies $T(\bar{x})$ in C) when $T(\bar{x})$ is a formula and $T(\bar{x})^C_\phi$ (the value of $T(\bar{x})$ in C at ϕ) when $T(\bar{x})$ is a term.

Let $\phi(^b_x)$, where b is in C, be the function for which $\phi(^b_x)(x) = b$ and $\phi(^b_x)(y) = \phi(y)$ for variables y different from x.

3.2.1 variables x, y, z... are terms

$$x^C_\phi = \phi(x)$$

3.2.2 $\perp$ is a formula

$C \models \perp[\phi]$ is false

3.2.3 If r, t are terms, then $r \epsilon t$ and $r = t$ are formulas

$C \models r \epsilon t\,[\phi]$ iff $r^C_\phi \in_C t^C_\phi$

$C \models r = t\,[\phi]$ iff $r^C_\phi = t^C_\phi$

3.2.4 If A, B are formulas, then A&B and $A \to B$ are formulas.

$C \models A \& B\,[\phi]$ iff $C \models A[\phi]$ and $C \models B[\phi]$

$C \models A \to B[\phi]$ iff $C \models A[\phi]$ implies $C \models B[\phi]$

3.2.5 If $A(x)$ is a formula, then $\forall x A(x)$ is a formula.

$$C \models \forall x A(x)[\phi] \text{ iff for all } b \text{ in } C \quad C \models A(x)[\phi(\tfrac{b}{x})]$$

3.2.6 If $A(x)$ is a formula and for all ϕ' there is an ordinal γ such that for all b in C, $C \models A(x)[\phi'(\tfrac{b}{x})]$ implies b is in C_γ then $\{x \mid A(x)\}$ is a term and $\{x \mid A(x)\}^C_\phi$ is the set in $C_{\gamma+1}$ of all b in C such that $C \models A(x)[\phi(\tfrac{b}{x})]$.

3.3 Some terms of SET

3.3.1 $\{x \mid x \neq x\}$ is a term. This is obvious.

3.3.2 If r, t are terms in which x is not free, then $\{x \mid x = r \vee x = t\}$, $\{x \mid \exists y (y \in r \,\&\, x \in y)\}$ and $\{x \mid \forall y (y \in x \rightarrow y \in r)\}$ are terms.

The reason for this is similar in all three cases, so let us take $\{x \mid \forall y (y \in x \rightarrow y \in r)\}$ as an example:

Take any ϕ. If $C \models \forall y (y \in x \rightarrow y \in r)[\phi(\tfrac{b}{x})]$, then b is clearly a subset of r^C_ϕ if r^C_ϕ is defined and a set which follows from the assumption that r is a term, so clearly there is a γ for ϕ with the properties we want, so $\{x \mid \forall y (y \in x \rightarrow y \in r)\}$ is then a term.

3.3.3 If $A(x)$ is a formula and t a term in which x is not free, then $\{x \mid x \in t \,\&\, A(x)\}$ is a term.

Follows by reasoning similar to 3.3.2.

3.3.4 $\{x \mid \forall_y (\emptyset \in y \,\&\, \forall_z (z \in y \rightarrow U\{z, \{z\}\} \in y) \rightarrow x \in y)\}$

is a term.

A bound for any ϕ in this case is obviously ω.

3.3.5 By applying stronger principles than those used so far we can find more terms that are of interest. For example terms corresponding to Fraenkel's substitution axiom:

$\{x \mid \exists y((y \in t \,\&\, A(y,x)) \,\&\, \forall u (A(y,u) \rightarrow u = x))\}$

given that t is a term in which x is not free.

3.4 By induction on the construction of $T(x)$ I will show that if $T(x)$ is a term (formula) of SET and t a term of SET, then $T(t)$ is a term (formula) of SET and if $t^C_\phi = b$, then

$T(x)^C_{\phi(\frac{b}{x})} = T(t)^C_\phi$ if $T(x)$ is a term of SET and

$C \models T(x)[\phi(\frac{b}{x})]$ iff $C \models T(t)[\phi]$ if $T(x)$ is a formula of SET.

3.4.1 $T(x) \equiv x$ obvious.

3.4.2 $T(x) \equiv \bot$ trivial. $T(x) \equiv z$ trivial.

3.4.3 $T(x) \equiv r(x) \in u(x)$.

It follows directly by IH that $r(t) \in u(t)$ is a formula, so

so assume $C \models r(x) \in u(x)[\phi(\frac{b}{x})]$. Then $r(x)^C_{\phi(\frac{b}{x})} \in_C u(x)^C_{\phi(\frac{b}{x})}$

By IH $r(x)^C_{\phi(\frac{b}{x})} = r(t)^C_\phi$, $u(x)^C_{\phi(\frac{b}{x})} = u(t)^C_\phi$,

so clearly $r(t)^C_\phi \in_C u(t)^C_\phi$, and so $C \models r(t) \in u(t)[\phi]$.

Similar reasoning in the other direction.

3.4.4 $T(x) \equiv r(x) = u(x)$.

Follows from IH by reasoning similar to 3.4.3.

3.4.5 The cases when $T(x)$ is on $\&, \rightarrow$ and $\forall$ -form follow in a rather uniform manner easily from IH. Let us take the $\forall$ -case as an example:
That $\forall z\, A(z,t)$ is a formula follows directly from IH. Now assume $C \vDash \forall z\, A(z,x)[\phi(\frac{b}{x})]$, so $C \vDash A(z,x)[\phi(\frac{b',b}{z,x})]$ for all b' in C. By IH $C \vDash A(z,t)[\phi(\frac{b'}{z})]$ for all b' in C, so $C \vDash \forall z\, A(z,t)[\phi]$. Similar reasoning in the other direction.

3.4.6 $T(x) \equiv \{z \mid A(z,x)\}$.

By IH $A(z,t)$ is a formula and $C \vDash A(z,x)[\phi(\frac{b}{x})]$ iff $C \vDash A(z,t)[\phi]$.

So assume $C \vDash A(z,t)[\phi(\frac{b'}{z})]$. By IH $C \vDash A(z,x)[\phi(\frac{b',b}{z,x})]$ so b' is in C_γ for a given γ since $\{z \mid A(z,x)\}$ is a term. So we have a bound for ϕ and $\{z \mid A(z,t)\}$ is a term.

To show $\{z \mid A(z,x)\}^C_{\phi(\frac{b}{x})} = \{z \mid A(z,t)\}^C_{\phi}$

assume b' is in $\{z \mid A(z,x)\}^C_{\phi(\frac{b}{x})}$, so

$C \vDash A(z,x)[\phi(\frac{b',b}{z,x})]$. By IH $C \vDash A(z,t)[\phi(\frac{b'}{z})]$ so b' is in $\{z \mid A(z,t)\}^C_{\phi}$. Similar reasoning in the other direction.

Throughout 3.4 we assume that no free variable in t becomes bound when t is substituted for x in $T(x)$.

3.5 By induction on the construction of $T(x)$ I will show that if

$C \models r=s[\phi]$, then $C \models T(r)[\phi]$ iff $C \models T(s)[\phi]$

if $T(x)$ is a formula and $T(r)^C_\phi = T(s)^C_\phi$ when $T(x)$ is a term.

3.5.1 $T(x) \equiv x$ obvious.

3.5.2 $T(x) \equiv \bot$ trivial, $T(x) \equiv z$ trivial.

3.5.3 $T(x) \equiv u(x) \in t(x)$.

By IH $u(r)^C_\phi = u(s)^C_\phi$, $t(r)^C_\phi = t(s)^C_\phi$,

so clearly $C \models u(r) \in t(r)[\phi]$ iff $C \models u(s) \in t(s)[\phi]$.

3.5.4 $T(x) \equiv u(x) = t(x)$.

Follows easily from IH by reasoning similar to 3.5.3.

3.5.5 As in 3.4.5 the cases when $T(x)$ is on $\&$, $\rightarrow$ and $\forall$ -form follow in a uniform manner using IH, so let us just take the $\forall$ -case as an example:

Assume $C \models \forall z\, A(z,r)[\phi]$, so $C \models A(z,r)[\phi(\frac{b}{z})]$

for all b in C. Now $C \models r = s[\phi(\frac{b}{z})]$, since we assume as in 3.4 that no free variables in r, s become bound when r, s are substituted for x in $T(x)$; so by IH $C \models A(z,s)[\phi(\frac{b}{z})]$,

so $C \models \forall z\, A(z,s)[\phi]$. Similar reasoning in the other direction.

3.5.6 $T(x) \equiv \{z \mid A(z,x)\}$.

To show $\{z \mid A(z,r)\}^C_\phi = \{z \mid A(z,s)\}^C_\phi$

assume b is in $\{z \mid A(z,r)\}^C_\phi$, so $C \vDash A(z,r)[\phi(\frac{b}{z})]$.

Now by IH $C \vDash A(z,s)[\phi(\frac{b}{z})]$, so clearly b is also in

$\{z \mid A(z,s)\}^C_\phi$. Similar reasoning in the other direction.

3.6 As a fragment of naive set theory we use the deductive apparatus of naive set theory as a formal system for SET. By induction on the length of deductions it is not hard to see that the rules of SET are sound for the semantic given in 3.2:

3.6.1 The basic case and the cases when deductions end with & I, & E, $\rightarrow$ I, $\rightarrow$ E, $\forall$ I and $\perp$ E-inferences offer no problems. It is only a matter of applying the IH.

3.6.2 $\mathcal{D}$ ends with $\forall$ E

$$\begin{array}{c} \vdots \\ \forall x\, A(x) \\ \hline A(t) \end{array}$$

By IH $C \vDash \forall x A(x)[\phi]$. To conclude $C \vDash A(t)[\phi]$

we have to know that if $t^C_\phi = b$ and $C \vDash A(x)[\phi(\frac{b}{x})]$,

then $C \vDash A(t)[\phi]$, but this follows from 3.4. so we are done.

3.6.3 $\mathcal{D}$ ends with $\in$ I

$$\frac{\begin{array}{c}\vdots\\ A(r)\end{array}}{r\in\{x\mid A(x)\}}$$

By IH $C\vDash A(r)[\phi]$, so using 3.4 we have $C\vDash A(x)[\phi(\frac{b}{x})]$ where $r^C_\phi = b$, so r^C_ϕ is in $\{x\mid A(x)\}^C_\phi$ which means

$C\vDash r\in\{x\mid A(x)\}[\phi]$, which is just what we want.

3.6.4 $\mathcal{D}$ ends with $\in$ E

$$\frac{\begin{array}{c}\vdots\\ r\in\{x\mid A(x)\}\end{array}}{A(r)}$$

By IH $C\vDash r\in\{x\mid A(x)\}[\phi]$, so $C\vDash A(x)[\phi(\frac{b}{x})]$ where $r^C_\phi = b$, so by 3.4 $C\vDash A(r)[\phi]$, which is what we want.

3.6.5 $\mathcal{D}$ ends with $=$ I

$$\frac{\begin{array}{cc}\not{x\in r} & \not{x\in s}\\ \vdots & \vdots\\ x\in s & x\in r\end{array}}{r=s}$$

By IH $C \models \forall x (x \in s \leftrightarrow x \in r)[\phi]$, so

$C \models x \in s \leftrightarrow x \in r[\phi(\frac{b}{x})]$

for all b, so clearly $s^C_\phi = r^C_\phi$, which means $C \models r = s[\phi]$.

3.6.6 $\mathcal{D}$ ends with $=E$

$$\frac{\begin{array}{cc} \vdots & \vdots \\ A(r) & r = s \end{array}}{A(s)}$$

By IH $C \models A(r)[\phi]$ and $C \models r = s[\phi]$, so using 3.5 we may conclude that $C \models A(s)[\phi]$, and so we are done.

3.7 To see that SET gives us an example of a wellfounded fragment of naive set theory we note first that clearly condition 1.1.1 of this chapter is satisfied. That also 1.1.2 is satisfied follows from 3.4. Using 3.6 we finally see that 1.1.3 too is satisfied, namely that E_{SET} is a wellfounded relation.

3.8 The construction of SET can of course be generalized in different ways. We could in the construction of C have used instead of the power-set operation any operation $\mathcal{S}$ such that $\mathcal{S}(x) \subseteq \mathcal{P}(x)$ and thereby construct higher order languages over other types of hierarchies, like Gödel's L for instance.
Instead of starting with $C_\emptyset \underset{\text{def.}}{=} \emptyset$, one could have started with any given first-order structure $\mathcal{A}$ and used a two-sorted language with variables and quantification over both individuals in $\mathcal{D}_\mathcal{A}$ and sets built up over $\mathcal{D}_\mathcal{A}$, thus obtaining a more general model theoretic language strong in expressive power.

3.9 As in section 2 of this chapter it is here easily seen how to obtain SET with v and $\exists$ as primitive constants and so we will have a language SET_I that will give us an example of a wellfounded fragment of naive set theory with intuitionistic logic.

4 Semantical motivation for the rules of contraction

4.1 Rules of inference and rules of contraction

We will here think of a system of natural deduction as providing a formal analysis of informal proofs in a given deductive practice.
The first step in such an analysis is the construction of a formal language in which we try to capture the language of the deductive practice on a syntactical level. Now to get a good analysis of the informal proofs, the rules of our system of natural deduction should be chosen so as to correspond as closely as possible to atomic steps in the informal proofs. If we explicate the semantical level of the deductive practice by giving a formal semantics (in the ordinary model theoretic sense) for the formal language, then we will be guided by certain criteria for the choice of rules: an atomic irreducible step corresponds naturally to the "... if..." or to the "... only if..." part of a clause in the semantics, which is to say that the validity of the step is based on just a clause in the semantics... on the very meaning of a certain constant.

The I- and E-rules are then thought to correspond respectively to the "... if..." part and to the "... only if..." part. So the pair of rules $(\underline{\Phi}, \underline{\Psi})$ for a given constant will be inverses of each other and the conclusion of a $\underline{\Psi}$ -inference should be derivable by elementary means from the premise deductions of the corresponding $\underline{\Phi}$ -inference and the minor deductions of the $\underline{\Psi}$ -inference.

In the ideal case where both $\underline{\Phi}$ and $\underline{\Psi}$ directly correspond to each direction in a given clause we should need to take just a premise deduction as it is. This simple operation will be called a simple operation, in the sense of chapter I section 2, of the first kind. It corresponds in the semantical level to:

$$C \vDash A[\phi] \Rightarrow C \vDash A[\phi].$$

In the $\rightarrow$ case we have a clause where the E-rule cannot in a natural way be in direct correspondence with the "... only if..." part of the $\rightarrow$clause. We need something more to handle this case. We have to be able to substitute a deduction $\mathcal{D}(\Gamma;A)$ for the open assumption A in $\mathcal{D}'(\{A,A_1,\ldots A_n\};B)$. This simple operation will be called a simple operation of the second kind. It corresponds on the semantical level to:

$$(C\models A[\phi]\ ,\ C\models A[\phi] \Rightarrow C\models B[\phi]) \Rightarrow C\models B[\phi].$$

In the $\forall$ -case there will be a need for still another operation. We have to be able to substitute a term t for x in $\mathcal{D}(x)$ at occurrences where x is free for substitution. This simple operation will be called a simple operation of the third kind. On the semantical level this will correspond to:

$$(C\models A(x)[\phi(\overset{b}{x})]\ ,\ t^{c}_{\phi}=b) \Rightarrow C\models A(t)[\phi].$$

Now a rule of contraction for $(\overline{\Phi}\,\Psi)$ -cuts will simply be an instruction telling us how to verify that the E-rule Ψ is an inversion of the I-rule Φ using certain very elementary operations. If the I-rules correspond closely enough to "... if..." parts of the semantics and the E-rules are derivable using only simple operations, then chains of contractions ought to be wellfounded if we are dealing with a welldefined semantics. If not we have a corresponding explanation on the semantical level which would not have a wellfounded basis. In view of the simple operations we consider this surely seems not to be in agreement with the assumption that the semantics is welldefined. So in this sense normalizability will be a kind of formal test of how close to the atomic steps our rules are, and consequently also a kind of formal test of how well our analysis captures the structure of the informal proofs.

The counter examples to normalizability given in section 5 of chapter I will now be interpreted in the following way:

The counter examples involving the $\in$ I and $\in$ -E-rules say that there is no semantics for the given formal language in which the $\in$ -clause looks like the $\in$ -rules.

The counter examples involving the $=$-I and $=$ E-rules tell us that there is no semantics for the given formal language in which $C \vdash A(r)[\phi]$ is derivable by elementary means from $C \vdash r = s\,[\phi]$ and $C \vdash A(s)[\phi]$ using the derived rules for "$\in$I" and "$\in$E".

The counter examples involving the $\perp$ E- rule say then simply that there is no semantics for the given formal language in which " $C \vdash A[\phi]$ or $C \nvdash A[\phi]$ " can be derived from atomic instances using the derived rules for " $\in$ I" and " $\in$ E".

In view of our interpretation of the counter examples to normalizability one may ask about the complexity involved in proving the $\in$, $=$ and $\perp$ -rules as properties of a given natural semantics for N. A good answer to this question will give us an idea of what kind of restrictions we have to put on the rules of contraction for $\in$, $=$ and $\perp$. We will here try to estimate this complexity by formulating a kind of subformula concept which is closely related to the I-rules and the reduction procedures used in the rules of contraction based on reduction properties.

4.2 Regularity of terms and formulas

4.2.1 Sub(A) and regular formulas

Let $A(x)$ be a formula and s, u terms. We will now define the (s,u)-subpairs of $(A(s), A(x))$:

If $A(x) \equiv \perp$, then $Sub_{(s,u)}(A(s), A(x)) \underset{def.}{=} \{(\perp, \perp)\}$.

If $A(x) \equiv r(x) \epsilon t(x)$ and $\not\vdash_N r(s) \epsilon t(s)$, then

$Sub_{(s,u)}(A(s), A(x)) =_{def.} \{(r(s) \epsilon t(s), r(x) \epsilon t(x))\}$.

If $A(x) \equiv r(x) \epsilon x$ and $\vdash_N r(s) \epsilon s$, then

$Sub_{(s,u)}(A(s), A(x)) =_{def.} \{(r(s) \epsilon u, r(x) \epsilon u)\}$.

If $A(x) \equiv r(x) \epsilon \{z \mid B(z,x)\}$ and $\vdash_N r(s) \epsilon \{z \mid B(z,s)\}$, then

$Sub_{(s,u)}(A(s), A(x)) =_{def.} \{(B(r(s),s), B(r(x),x))\}$.

If $A(x) \equiv r(x) = t(x)$, then $Sub_{(s,u)}(A(s), A(x)) =_{def.}$

$\{(z \epsilon r(s), z \epsilon r(x)), (z \epsilon t(s), z \epsilon t(x)), (z \epsilon r(u), z \epsilon r(x)), (z \epsilon t(u), z \epsilon t(x))\}$.

If $A(x) \equiv B(x) \& C(x)$, then $Sub_{(s,u)}(A(s), A(x)) =_{def.}$

$\{(B(s), B(x)), (C(s), C(x))\}$.

If $A(x) \equiv B(x) \rightarrow C(x)$, then $Sub_{(s,u)}(A(s), A(x)) =_{def.}$

$\{(B(u), B(x)), (C(s), C(x))\}$.

If $A(x) \equiv \forall z B(z,x)$, then $Sub_{(s,u)}(A(s), A(x)) =_{def.}$

$\{(B(z,s), B(z,x))\}$.

If x is not free in A(x) we simply write Sub(A) for $Sub_{(s,u)}(A(s), A(x))$.

A <u>(s,u)-subchain</u> of $(A(s), A(x))$ is a sequence $P_1 \ldots P_n \ldots$ of pairs of formulas such that $P_1 \equiv (A(s), A(x))$ and for all i P_{i+1} is in $Sub_{(s,u)}(P_i)$ or $Sub_{(u,s)}(P_i)$. Such a subchain is <u>proper</u> if $P_{i+1} \not\equiv P_i$ for all i.

A formula A(x) is said to be <u>(s=u)-regular</u> if all proper (s,u)-subchains of $(A(s), A(x))$ and all proper (u,s)-subchains of $(A(u), A(x))$ are finite.

4.2.2 <u>Regular terms</u>

Let s, u be terms such that $\vdash_N s = u$.

Let $\widetilde{t}$, for a given term t, be the collection of terms r such that $\vdash_N r=t$. By E_N-induction on t we will define what it means for terms t(x), where t(s) and t(u) are in $\widetilde{t}$, to be (s=u)-regular:

If t has no predecessors, then t(x) is (s=u)-regular.

If t has some predecessors, then t(x) is (s=u)-regular if $r(x)\in t(x)$ is (s=u)-regular for all r(x) where $E_N(r(s),t(s))$ and r(x) is (s=u)-regular.

4.2.3 Let $T(\bar{x})$ be a term or formula and s, u terms such that $\vdash_N s=u$.

If $\bar{t}(x)\equiv t_1(x)\ldots t_n(x)$ are (s=u)-regular, then $T(\bar{t}(x))$ is (s=u)-regular.

This will be proved by induction on the construction of $T(\bar{x})$:

4.2.3.1 $T(\bar{x})\equiv x_i$. By assumption $t_i(x)$ is (s=u)-regular.

4.2.3.2 The case when $T(\bar{x})\equiv\bot$ is trivial and the cases when $T(\bar{x})$ is a formula on & , $\rightarrow$ or $\forall$ -form follow directly using IH. Let us just check the $\rightarrow$ case as an example:

By IH $A(\bar{t}(x))$ and $B(\bar{t}(x))$ are (s=u)-regular, so clearly all proper (s,u) - subchains of $(A(\bar{t}(s))\rightarrow B(\bar{t}(s)),\ A(\bar{t}(x))\rightarrow B(\bar{t}(x)))$ and all proper (s,u)-subchains of $(A(\bar{t}(u))\rightarrow B(\bar{t}(u)),\ A(\bar{t}(x))\rightarrow B(\bar{t}(x)))$ will be finite which means $A(\bar{t}(x))\rightarrow B(\bar{t}(x))$ is (s=u)-regular.

4.2.3.3 $T(\bar{x})\equiv r(\bar{x})=t(\bar{x})$.

Now $\not\vdash_N z\in v$ for any v following the wellfoundedness of E_N so $(z\in r(\bar{t}(s)),\ z\in r(\bar{t}(x)))$, $(z\in t(\bar{t}(s)),\ z\in t(\bar{t}(x)))$, $(z\in r(\bar{t}(u)),\ z\in r(\bar{t}(x)))$, $(z\in t(\bar{t}(u)),\ z\in t(\bar{t}(x)))$ will have no proper (s,u) or (u,s) -subpairs. This means $r(\bar{t}(x))=t(\bar{t}(x))$ is (s=u)-regular.

4.2.3.4 $T(\bar{x}) \equiv r(\bar{x}) \in t(\bar{x})$.

If $\not\vdash_N r(\bar{t}(s)) \in t(\bar{t}(s))$ we are immediately done, so assume $\vdash_N r(\bar{t}(s)) \in t(\bar{t}(s))$.
By IH we know that $r(\bar{t}(x))$ and $t(\bar{t}(x))$ are $(s=u)$-regular, so by definition of $(s=u)$-regularity for terms it follows that $r(\bar{t}(x)) \in t(\bar{t}(x))$ is $(s=u)$-regular.

4.2.3.5 $T(\bar{x}) \equiv \{z \mid A(z,\bar{x})\}$.

If $\{z \mid A(z,\bar{t}(s))\}$ has no E_N-predecessors $\{z \mid A(z,\bar{t}(x))\}$ is $(s=u)$-regular by definition. So assume $\{z \mid A(z,\bar{t}(s))\}$ has some E_N-predecessors:

If $\vdash_N r(s) \in \{z \mid A(z,\bar{t}(s))\}$ we want to establish that $r(x) \in \{z \mid A(z,\bar{t}(x))\}$ is $(s=u)$-regular if $r(x)$ is $(s=u)$-regular.

By IH $A(r(x),\bar{t}(x))$ is $(s=u)$-regular then, so all proper (s,u)-subchains of $(A(r(s),\bar{t}(s)), A(r(x),\bar{t}(x)))$ and all proper (u,s)-subchains of $(A(r(u),\bar{t}(u)), A(r(x),\bar{t}(x)))$ are finite. This means clearly that $r(x) \in \{z \mid A(z,\bar{t}(x))\}$ is $(s=u)$-regular.

4.2.4 All terms $t(x)$ and formulas $A(x)$ are $(s=u)$-regular.

4.2.4.1 x is $(s=u)$-regular.

Assume $E_N(r(s),s)$ and $r(x)$ is $(s=u)$-regular.
Variables $x_i \not\equiv x$ are $(s=u)$-regular since x_i has no E_N-predecessors.
Let $T(\bar{x}) \equiv x_i \in u(\bar{x})$, then it follows from 4.2.3 that

$r(x) \in u(\bar{x})$ is $(s=u)$-regular. So proper (s,u)-subchains of $(r(s)\in u,\ r(x)\in u)$ are finite.

Put $T(\bar{x}) \equiv x_1 \in s(\bar{x})$, then likewise by 4.2.3 $r(x)\in s(\bar{x})$ is $(s=u)$-regular, so proper (u,s)-subchains of $(r(u)\in s,\ r(x)\in s)$ are finite.

All this means $r(x)\in x$ is $(s=u)$-regular, so x is $(s=u)$-regular.

4.2.4.2 $t(x)$ and $A(x)$ are $(s=u)$-regular for all $t(x)$ and $A(x)$.

This is an easy consequence of 4.2.3 and 4.2.4.1.

4.2.5 $\rho(A)$

By 4.2.4 we see that $A(x)$ is $(s=u)$-regular for all $A(x)$.
We will then define the $(s=u)$-rank of $(A(s), A(x))$,

$\rho_{(s=u)}(A(s), A(x))$, to be the maximal length of a (s,u)-subchain of $(A(s), A(x))$. When x is not free in $A(x)$ we simply write $\rho(A)$ and read this as the rank of A.

4.2.5.1 It is easily seen that we have the following properties of the rank notion:

$\rho_{(s=u)}(\bot) = 1.$

$\rho_{(s=u)}(r(s)\in t(s),\ r(x)\in t(x)) = 1$ if $\not\vdash_N r(s)\in t(s).$

$\rho_{(s=u)}(A(s), A(x)),\ \rho_{(s=u)}(B(s), B(x)) < \rho_{(s=u)}(A(s)\&B(s), A(x)\&B(x)).$

$\rho_{(s=u)}(A(u), A(x)),\ \rho_{(s=u)}(B(s), B(x)) < \rho_{(s=u)}(A(s)\to B(s), A(x)\to B(x)).$

$\rho_{(s=u)}(A(z,s), A(z,x)) < \rho_{(s=u)}(\forall z\, A(z,s), \forall z\, A(z,x)).$

$\rho_{(s=u)}(z\in r(s), z\in r(x)),\ \rho_{(s=u)}(z\in r(u), z\in r(x)),$

$\rho_{(s=u)}(z\in t(s), z\in t(x)),\ \rho_{(s=u)}(z\in t(u), z\in t(x)) <$

$< \rho_{(s=u)}(r(s)=t(s),\ r(x)=t(x)).$

$$\rho_{(s=u)}\big(A(r(s),s)\,,\,A(r(x),x)\big) < \rho_{(s=u)}\big(r(s)\in\{z|A(z,s)\},r(x)\in\{z|A(z,x)\}\big)$$

if $\vdash_N r(s)\in\{z|A(z,s)\}$.

$$\rho_{(s=u)}\big(r(s)\in u\,,\,r(x)\in u\big) < \rho_{(s=u)}\big(r(s)\in s\,,\,r(x)\in x\big)$$

if $\vdash_N r(s)\in S$.

4.2.6 For fragments N with intuitionistic logic we make the following additions:

$$Sub_{(s,u)}\big(A(s)\vee B(s)\,,\,A(x)\vee B(x)\big) =_{def.} \{(A(s),A(x))\,,(B(s),B(x))\}.$$

$$Sub_{(s,u)}\big(\exists z\,A(z,s)\,,\,\exists z A(z,x)\big) =_{def.} \{(A(z,s)\,,\,A(z,x))\}.$$

It is then not hard to see that 4.2.3 and 4.2.4 still hold with these additions.
This subformula concept does not provide a close correspondence with the $\exists$I-rule, but rather with the $\exists$E-rule and this is perhaps as it should be in a certain sense: namely that the $\exists$-rules are a kind of inverse of the $\forall$-rules. But comparing with clauses in the semantics this just reflects that the $\exists$I-rule does not directly correspond to the "... if..." part of the $\exists$-clause.

4.3 Motivations

Let $\frac{}{\Phi}$ be a rule of inference in N. We will here identify $\frac{}{\Phi}$ with the collection of deductions $\mathcal{D}$ on $\frac{}{\Phi}$-form. So we may talk about subrules $\frac{}{\Phi'}\subseteq\frac{}{\Phi}$ etc...

An I-rule $\frac{}{\Phi}$ is ρ-compatible at $\frac{}{\Phi'}\subseteq\frac{}{\Phi}$ if for all $\mathcal{D}$ in $\frac{}{\Phi'}$ both premises and assumptions, if any, cancelled at the last inference in $\mathcal{D}$ are of lower rank than the conclusion of $\mathcal{D}$.

An E-rule $\bar{\Psi}$ is locally derivable from the I-rule $\bar{\Phi}$ at $\bar{\Psi}' \subseteq \bar{\Psi}$ if $\bar{\Psi}'$ is an inversion of $\bar{\Phi}$.

4.3.1 First motivation

A $(\bar{\Phi},\bar{\Psi})$ -cut is a cut where the deduction of the cut is in $\bar{\Phi}$ and also a major deduction of an $\bar{\Psi}$ -inference.

Let $\bar{\Phi}' \subseteq \bar{\Phi}$ and $\bar{\Psi}' \subseteq \bar{\Psi}$.

We may define a rule of contraction for $(\bar{\Phi}',\bar{\Psi}')$ -cuts when $\bar{\Phi}$ is ρ -compatible at $\bar{\Phi}'$ and $\bar{\Psi}$ is locally derivable from $\bar{\Phi}$ at $\bar{\Psi}'$ using only simple operations of the first and second kind.

4.3.1.1 & I is ρ -compatible at & I and & E is locally derivable from & I at & E using only simple operations of the first kind. This motivates that the & -rule of contraction is always defined for $(\&I,\&E)$ -cuts.

4.3.1.2 $\rightarrow$ I is ρ -compatible at $\rightarrow$ I and $\rightarrow$ E is locally derivable from $\rightarrow$ I at $\rightarrow$ E using only simple operations of the first and second kind. This motivates that the $\rightarrow$ -rule of contraction is always defined for $(\rightarrow I,\rightarrow E)$ -cuts.

4.3.1.3 Let $(\epsilon I)_{\mathcal{C}}$ be the collection of deductions $\mathcal{D}$ on ϵ I-form such that $\mathcal{C}(\mathcal{D})$.
ϵ I is ρ -compatible at $(\epsilon I)_{\mathcal{C}}$ and ϵ E is locally derivable from ϵ I at $(\epsilon E)_{\mathcal{C}}$ using only simple operations of the first kind. This motivates that the ϵ -rule of contraction is defined for $((\epsilon I)_{\mathcal{C}},(\epsilon E)_{\mathcal{C}})$ -cuts.

This is also the largest subrule of ϵ I, ϵ E we can motivate on basis of the rank notion.

4.3.1.4 Consider the fragment $N_{\{\&,\to,\epsilon\}}$ of N based on $\&$, $\to$ and ϵ
Using the rank notion it is easily seen that every deduction can be transformed into a $\mathcal{C}$ -normal deduction using the rules of contraction:

By a $\mathcal{C}$ -cut in $\mathcal{D}$ we will understand a cut in $\mathcal{D}$ for which there is a rule of contraction defined.
Let $d(\mathcal{D})$ be the maximal rank of $\mathcal{C}$ -cut in $\mathcal{D}$ and let $l(\mathcal{D})$ be the number of $\mathcal{C}$ -cuts in $\mathcal{D}$ with $d(\mathcal{D})$ -rank. We order pairs $(d(\mathcal{D}), l(\mathcal{D}))$ lexicographically.
Choose a $\mathcal{C}$ -cut A in $\mathcal{D}$ such that the cut has rank $d(\mathcal{D})$ and there are no $\mathcal{C}$ -cuts of rank $d(\mathcal{D})$ in the deduction of the cut A or in the minor deductions, if any, at the cut.
Now by inspection of the rules of contraction it is easily seen that $(d(\mathcal{D}'), l(\mathcal{D}')) < (d(\mathcal{D}), l(\mathcal{D}))$
where $\mathcal{D}'$ is the contraction of $\mathcal{D}$ at the cut A.
Proceeding in this manner we will finally reach a $\mathcal{C}$ -normal deduction.

4.3.2 Second motivation

We may define a rule of contraction for (Φ', Ψ') -cuts when Φ is ρ -compatible at Φ' and Ψ is locally derivable from Φ at Ψ'.

4.3.2.1 $\forall$ I is ρ -compatible at $\forall$ I and $\forall$ E is locally derivable from $\forall$ I at $\forall$ E. This motivates that the $\forall$ -rule of contraction is always defined for $(\forall I, \forall E)$ -cuts.

4.3.2.2 $=$ I is ρ -compatible at $=$ I.
So if $\Psi' \subseteq =E$ is given such that $=$E is locally derivable from $=$I at Ψ' we may define $=$ -contractions for $(=I, \Psi')$ -cuts.
This motivates that we have no restrictions on contractions for such cuts.

4.3.3 Third motivation

Here we will deal with rules of contraction based on reduction properties. The idea is that we want to reduce the complexity of applications of a certain E-rule Ψ using the I-rule for a given formula. We will interpret this reduction of complexity as a reduction in rank complexity of new applications of Ψ.

4.3.3.1 The $\bot$ E-rule

We may define a rule of contraction at a $\bot$ E-cut whenever the conclusions of new applications of the $\bot$ E-rule in the contraction are of lower rank than the given cut.

So it is easily seen that it is only in the ϵ -case there is a motivation for some restrictions:

Given

$$\left.\begin{array}{c} \neg\, r\epsilon\{x|A(x)\} \\ \vdots \\ \bot \\ \hline r\epsilon\{x|A(x)\} \end{array}\right\} \mathcal{D}$$

we may define a contraction here only when $\mathcal{C}(\mathcal{D})$ since $\rho(A(r)) < \rho(r\epsilon\{x|A(x)\})$ follows if $\vdash_N r\epsilon\{x|A(x)\}$ and in other cases the rank notion will tell us nothing.

4.3.3.2 The $=$-cuts

Now let $(=E)$ be a subrule of $=E$ such that a $(=I,(=E))$-cut will have no rule of contraction motivated by the second motivation.
We may define a rule of contraction at a $(=I,(=E))$-cut if minor premises of new applications of the $=$E-rule in

the contraction are all of lower $(s=u)$-rank than the minor premise at the given cut and the deduction of the cut satisfies $\mathcal{C}$. So let $\mathcal{D}_{s,u}(\Gamma; s=u)$ be a deduction on I-form satisfying $\mathcal{C}$;

4.3.3.2.1 $A(x) \equiv A_1(x) \& A_2(x)$.

Minor premises in new applications of the $=$E-rule in the contraction will be $A_1(s)$ and $A_2(s)$.

$$\rho_{(s=u)}\left(A_i(s), A_i(x)\right)_{i=1,2} < \rho_{(s=u)}\left(A_1(s)\&A_2(s), A_1(x)\&A_2(x)\right).$$

This motivates then that we have a rule of contraction defined in this case.

4.3.3.2.2 $A(x) \equiv A_1(x) \to A_2(x)$.

Minor premises in new applications of the $=$E-rule in the contraction will be $A_1(u)$ and $A_2(s)$.

$$\rho_{(s=u)}\left(A_1(u), A_1(x)\right), \rho_{(s=u)}\left(A_2(s), A_2(x)\right) <$$

$$< \rho_{(s=u)}\left(A_1(s) \to A_2(s), A_1(x) \to A_2(x)\right).$$

This motivates that we have a rule of contraction defined in this case.

4.3.3.2.3 $A(x) \equiv \forall z B(z,x)$.

We will have $B(z,s)$ as a minor premise in the new application of the $=$E-rule in the contraction.

$$\rho_{(s=u)}\left(B(z,s), B(z,x)\right) < \rho_{(s=u)}\left(\forall z B(z,s), \forall z B(z,x)\right).$$

This will motivate that we define a rule of contraction in this case.

4.3.3.2.4 $A(x) \equiv t(x) \in \{z \mid B(z,x)\}$.

$B(t(s),s)$ will be minor premise in the new application of the $=$E-rule in the contraction.

$\rho_{(s=u)}(B(t(s),s), B(t(x),x)) < \rho_{(s=u)}(t(s) \in \{z \mid B(z,s)\}, t(x) \in \{z \mid B(z,x)\})$

if $\vdash_N t(s) \in \{z \mid B(z,s)\}$.

So a rule of contraction may be defined when the minor deduction at the given cut satisfies $\mathcal{C}$. This motivates the restriction enforced on contractions in this case.

4.3.3.2.5 $A(x) \equiv t(x) \in X$.

Here we have a combination of reductions and simple operations of the third kind. $t(s) \in u$ will be minor premise in the new application of the $=$E-rule in the contraction.

$\rho_{(s=u)}(t(s) \in u, t(x) \in u) < \rho_{(s=u)}(t(s) \in s, t(x) \in x)$.

if $\vdash_N r(s) \in S$.

So just as in the previous case we may define a rule of contraction when the minor deduction at the given cut satisfies $\mathcal{C}$. We have then a motivation for the restrictions on contractions in this case.

4.3.3.2.6 $A(x) \equiv t(x) \in z$.

Clearly there will never be a motivation based on the rank notion for a rule of contraction in this case since $\not\vdash_N t(s) \in z$ by the wellfoundedness of E_N.

4.3.3.2.7 $A(x) \equiv t(x) = v(x)$.

Minor premises in new applications of the $=$E-rule in the contraction will be $z \in t(s)$, $z \in v(s)$, $z \in t(u)$, $z \in v(u)$.

$$\rho_{(s=u)}(z\epsilon t(s), z\epsilon t(x)) \ , \ \rho_{(s=u)}(z\epsilon v(s), z\epsilon v(x)) \ , \ \rho_{(s=u)}(z\epsilon t(u), z\epsilon t(x)),$$

$$\rho_{(s=u)}(z\epsilon v(u), z\epsilon v(x)) < \rho_{(s=u)}(t(s)=v(s) \ , \ t(x)=v(x)) .$$

So a rule of contraction will always be motivated in this case.

4.3.3.3 The motivation given for $=$-contractions is based on the assumption that $\mathcal{D}_{s,u}$ satisfies $\mathcal{C}$. In defining the rules of contraction we do not strictly follow this. It is not hard to see that we only have to demand that $\mathcal{D}_{s,u}$ satisfies $\mathcal{C}$ in those cases where we have to enforce a restriction based also on the minor deduction at the cut. So we skip the restriction on $\mathcal{D}_{s,u}$ in the $\&$, $\rightarrow$ and $\forall$-cases. This is a more pragmatic motivation, but using it we analyze a little more of the rules.

4.3.4 For wellfounded fragments N with intuitionistic logic it is not hard to see that the first motivation gives us a motivation for always defining a rule of contraction for $(\vee I, \vee E)$-cuts. Contractions at $(\exists I, \exists E)$-cuts do not really have a motivation in terms of the second motivation since $\exists I$ is not ρ-compatible at any reasonable subrule. We have to extend the analysis to cover this case, but for the time being we note that the assumption cancelled at a $\exists E$-inference is of lower rank than the major premise.

Once the motivations for the rules of contraction at $(\vee I, \vee E)$ and $(\exists I, \exists E)$-cuts are clear it is not difficult to see that contractions at $=$-cuts with minor premises on $\vee$ or $\exists$-form are always motivated.

4.4 Notes

The particular formulation of rank for formulas given in this section is inspired by Schütte's notion of rank for terms and formulas in simple type theory. (See [Sch]).

The idea of looking at the normalization of proofs in natural deduction as connected with semantical principles is to a great extent inspired by work of Martin-Löf and of Prawitz. In a joint seminar on meaning theory a couple of years ago they presented ideas connected with natural deduction and semantics which have been a constant source of inspiration to me when thinking about these matters.
(See also [M4], [Pr3], [Pr4].)

The proof sketched in 4.3.1.4 follows Prawitz proof of normalization in first-order logic. (See [Pr1] pp 39-41 and 49-51.)

III $\mathcal{C}$-NORMALIZABILITY OF DEDUCTIONS IN WELLFOUNDED FRAGMENTS OF NAIVE SET THEORY

In this chapter we will prove that deductions in a wellfounded fragment of naive set theory are $\mathcal{C}$-normalizable. It may be useful to compare this proof with what we did in section 3 of chapter II when we constructed the language SET and provided it with a semantics.

In section 1 of the present chapter we will build up a counterpart to the structure C by defining the W-objects, W-predicates and give meaning to equality and membership in this universe.

In section 2 we will define a sort of semantics $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ for deductions $\mathcal{D}(\Gamma; A)$ based on the structure we build up in section 1. We will also prove counterparts to "$C \models s=u[\phi]$, $C \models A(s)[\phi] \Rightarrow C \models A(u)[\phi]$" and "$C \models A[\phi]$ or $C \not\models A[\phi]$" as properties of the given semantics and that if a deduction $\mathcal{D}(\Gamma; A)$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$, then $\mathcal{D}$ is $\mathcal{C}$-normalizable.

In section 3 we will prove a counterpart to "$C \models A(x)[\phi(\frac{b}{x})]$, $b = t^C_\phi \Rightarrow C \models A(t)[\phi]$".

And finally in section 4 we will prove that the rules of inference of N are sound relative to the given semantics. Now since assumptions A by definition will satisfy $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ this implies that all deductions in N are $\mathcal{C}$-normalizable. And just as in the proof that the rules of inference are sound relative to the semantics for SET, we will have to use the properties of $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ proved in sections 2 and 3 to have this proof go through.

1 Wellfoundedness predicates and wellfoundedness objects

1.1 Let A be a formula. A W-predicate (W for wellfoundedness) α_A of type A is a predicate, defined for deductions $\mathcal{D}(\Gamma;A)$, such that the following three conditions hold:

1.1.1 The deduction consisting only of the assumption A satisfies α_A.

1.1.2 If $\mathcal{D}$ satisfies α_A, then $\mathcal{D}$ is $\mathcal{C}$-normalizable.

1.1.3 If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies α_A iff either $\mathcal{D}$ has a $\mathcal{C}$-main cut and the contraction of $\mathcal{D}$ at this cut satisfies α_A, or else $\mathcal{D}$ has no $\mathcal{C}$-main cut, all minor and major deductions along the $\mathcal{C}$-main branch are $\mathcal{C}$-normalizable, and the $\mathcal{C}$-main deduction, if there is one, is also $\mathcal{C}$-normalizable.

The condition given in 1.1.3 will occur frequently in what follows for different predicates, so let us refer to it as $\mathcal{D}$ has the E(P)-property for a given predicate P defined for deductions. 1.1.3 is then: if $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies α_A iff $\mathcal{D}$ has the $E(\alpha_A)$-property.

1.2 A W-predicate α_A is closed under

negation iff for all

$$\left.\begin{array}{c} \neg A \\ \vdots \\ \bot \\ \hline A \end{array}\right\} \mathcal{D}$$

$\mathcal{D}$ satisfies α_A if:

$\neg A$ $\vdots$ $\bot$ is $\mathcal{C}$-normalizable and

$$\begin{array}{c} A \\ \vdots \\ \bot \\ \hline \neg A \end{array} \qquad \text{is } \mathcal{C}\text{-normalizable and} \qquad \vdots \ \bot$$

is $\mathcal{C}$-normalizable for all $\dfrac{\begin{array}{c}\not{A}\\ \vdots\\ \bot\end{array}}{\neg A}$

such that $\begin{array}{c}A\\ \vdots\\ \bot\end{array}$ is $\mathcal{C}$-normalizable for all $\vdots\ A$ satisfying α_A.

As in 1.1.3 we here have a condition that will occur a number of times in what follows, so for a given predicate P defined for deductions let us refer to as $\mathcal{D}$ has the $\bot$(P)-property. 1.2 is then: A W-predicate α_A is closed under negation

if for all $\left.\dfrac{\begin{array}{c}\not{\neg A}\\ \vdots\\ \bot\end{array}}{A}\right\}\mathcal{D}$, $\mathcal{D}$ satisfies α_A if $\mathcal{D}$ has the $\bot(\alpha_A)$-property.

1.3 Assume $\vdash_N s=u$ and let $\mathcal{D}_{s,u}(\Gamma; s=u)$ be a deduction on I-form

$$\frac{\begin{array}{cc}\not{x\in s} & \not{x\in u}\\ \vdots & \vdots\\ x\in u & x\in s\end{array}}{s=u}$$

$$\begin{array}{cc} r\in s & r\in u\\ \vdots & \vdots\end{array}$$

such that $r\in u$ and $r\in s$ are $\mathcal{C}$-normalizable for all terms r,

where these deductions are obtained by substituting r for x in the premise deductions of $\mathcal{D}_{s,u}$. Thus in particular $\mathcal{D}_{s,u}$ has to be $\mathfrak{C}$-normalizable.

For formulas A(x) and W-predicates $\alpha_{A(s)}$, $\alpha_{A(u)}$ of types A(s), A(u), we define

$$\alpha_{A(s)} = \alpha_{A(u)}(\mathcal{D}_{s,u}, A(x))$$

(read as: $\alpha_{A(s)}$ is equal to $\alpha_{A(u)}$ modulo $\mathcal{D}_{s,u}$ and A(x)) to hold iff for all deductions $\mathcal{D}_1(\Gamma_1; A(s))$ and

$\mathcal{D}_2(\Gamma_2; A(u))$, if $\mathcal{D}_1$ satisfies $\alpha_{A(s)}$, then

$$\left(A(x)\right)\frac{\mathcal{D}_1 \qquad \mathcal{D}_{s,u}}{A(u)}$$

satisfies $\alpha_{A(u)}$, and if $\mathcal{D}_2$ satisfies $\alpha_{A(u)}$, then

$$\left(A(x)\right)\frac{\mathcal{D}_2 \qquad \mathcal{D}_{s,u}}{A(s)}$$

satisfies $\alpha_{A(s)}$.

1.4 For any term t, let $\tilde{t}$ be the collection of all terms r such that $\vdash_N r = t$.

In 1.4.1 - 1.4.3 it will be assumed that r is a term in $\tilde{t}$ and s, u, t(x) terms such that t(s) and t(u) belong to $\tilde{t}$.

By E_N induction on t, I will define the <u>negation closed W-objects of type r</u> (using α_r as a symbol for such objects), and also define the relation

$$\alpha_{t(s)} = \alpha_{t(u)}(\mathcal{D}_{s,u}, t(x))$$

(read as: $\alpha_{t(s)}$ is equal to $\alpha_{t(u)}$ modulo $\mathcal{D}_{s,u}$ and $t(x)$)

where $\vdash_N s=u$ and $\mathcal{D}_{s,u}$ is a deduction satisfying the conditions stated in 1.3.

1.4.1 If t has no E_N predecessor, then the term r itself is the only negation closed W-object of type r.

1.4.1.1 $\alpha_{t(s)} = \alpha_{t(u)}(\mathcal{D}_{s,u}, t(x))$

holds by definition if t has no E_N predecessors.

1.4.2 If t does have E_N predecessors, then a negation closed W-object of type r is a function defined for negation closed W-objects α_v of type v, where $E_N(v,r)$ such that for every such α_v $\alpha_r(\alpha_v)$ is a negation closed W-predicate of type $v \in r$ and furthermore for all v(x),s,u and $\mathcal{D}_{s,u}$ such that $E_N(v(s), r)$ and $E_N(v(u), r)$, and all negation closed W-objects $\alpha_{v(s)}$, $\alpha_{v(u)}$ if $\alpha_{v(s)} = \alpha_{v(u)}(\mathcal{D}_{s,u}, v(x))$ then

$$\alpha_r(\alpha_{v(s)}) = \alpha_r(\alpha_{v(u)})(\mathcal{D}_{s,u}, v(x) \in r).$$

1.4.2.1 The definition of the equality relation has two different clauses if t has E_N predecessors. First assume that $t(x) \not\equiv x$. Then $\alpha_{t(s)} = \alpha_{t(u)}(\mathcal{D}_{s,u}, t(x))$ holds iff

$$\alpha_{t(s)}(\alpha_{v(s)}) = \alpha_{t(u)}(\alpha_{v(u)})(\mathcal{D}_{s,u}, v(x) \in t(x))$$

for all terms v(x) and negation closed W-objects $\alpha_{v(s)}, \alpha_{v(u)}$ in the domain of $\alpha_{t(s)}$ such that $\alpha_{v(s)} = \alpha_{v(u)}(\mathcal{D}_{s,u}, v(x))$.

Note that the condition on α_r stated in 1.4.2 is thus equivalent to: $\alpha_r = \alpha_r(\mathcal{D}_{s,u}, r)$ for all $\mathcal{D}_{s,u}$.

1.4.2.2 In the case $t(x) \equiv x$, $\alpha_s = \alpha_u(\mathcal{D}_{s,u}, x)$ is defined to hold iff for all negation closed W-objects α_v in the domain of α_s

$$\begin{array}{l} \vdots \\ v \in S \\ \vdots \\ v \in u \text{ satisfies } \alpha_u(\alpha_v) \text{ if } v \in S \text{ satisfies } \alpha_s(\alpha_v) \end{array} \qquad \begin{array}{l} \vdots \\ \\ \vdots \\ \\ \vdots \end{array}$$

and also

$$\begin{array}{l} \vdots \\ v \in u \\ \vdots \\ v \in S \text{ satisfies } \alpha_s(\alpha_v) \text{ if } v \in u \text{ satisfies } \alpha_u(\alpha_v). \end{array} \qquad \begin{array}{l} \vdots \\ \\ \vdots \\ \\ \vdots \end{array}$$

The long deductions indicated above are obtained by substituting v for x and deductions of $v \in S$ respectively $v \in u$ for the open assumptions in the premise deductions of $\mathcal{D}_{s,u}$.

2 $\underline{\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}})}$.

Let $T(\bar{x})$ be a term or formula, $\bar{t} = t_1 \ldots t_n$ terms and $\bar{\alpha}_{\bar{t}} = \alpha_{t_1} \ldots \alpha_{t_n}$ negation closed W-objects of types $\bar{t}$. By induction on the construction of $T(\bar{x})$ we will define $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}})$, show that $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a negation closed W-predicate of type $T(\bar{t})$ if $T(\bar{x})$ is a formula and a negation closed W-object of type $T(\bar{t})$ if $T(\bar{x})$ is a term and also show

that if $\alpha_{t_i} = \alpha'_{t'_i}(\mathcal{D}_{S,u}, t_i(x))$ for $1 \leq i \leq n$,

then $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{T(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{S,u}, T(t_1(x) \ldots t_n(x)))$

where $\bar{t} = t_1(s) \ldots t_n(s)$, $\bar{t}' = t_1(u) \ldots t_n(u)$

and $\bar{\alpha}_{\bar{t}}$, $\bar{\alpha}'_{\bar{t}'}$, are negation closed W-objects of types $\bar{t}$ and $\bar{t}'$ respectively.

2.1 $T(\bar{x}) \equiv x_i$.

$\phi_{x_i}(\bar{\alpha}_{\bar{t}}) =_{def.} \alpha_{t_i}$, which by assumption is a negation closed W-object of type t_i.

Clearly $\phi_{x_i}(\bar{\alpha}_{\bar{t}}) = \phi_{x_i}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{S,u}, t_i(x))$ if $\alpha_{t_i} = \alpha'_{t'_i}(\mathcal{D}_{S,u}, t_i(x))$ since $\phi_{x_i}(\bar{\alpha}_{\bar{t}}) =_{def.} \alpha_{t_i}$ and $\phi_{x_i}(\bar{\alpha}'_{\bar{t}'}) =_{def.} \alpha'_{t'_i}$.

2.2 $T(\bar{x}) \equiv \perp$.

A deduction $\mathcal{D}(\Gamma; \perp)$ satisfies $\phi_{\perp}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ is $\mathcal{C}$-normalizable.

2.2.1 Clearly $\phi_{\perp}(\bar{\alpha}_{\bar{t}})$ is a W-predicate of type $\perp$. If a deduction

$$\left.\begin{array}{c} \not\neg \\ \vdots \\ \perp \\ \hline \perp \end{array}\right\} \mathcal{D}$$

has the $\perp(\phi_{\perp}(\bar{\alpha}_{\bar{t}}))$-property it obviously satisfies $\phi_{\perp}(\bar{\alpha}_{\bar{t}})$, so $\phi_{\perp}(\bar{\alpha}_{\bar{t}})$ is a negation closed W-predicate.

Trivially $\phi_{\perp}(\bar{\alpha}_{\bar{t}}) = \phi_{\perp}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{s,u}, \perp)$ holds.

2.3 $\Gamma(\bar{x}) \equiv r(\bar{x}) \in t(\bar{x})$.

By IH $\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ are negation closed W-objects of type $r(\bar{t})$ and $t(\bar{t})$ respectively. Using this we define $\phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ as follows:

If $\vdash_N r(\bar{t}) \in t(\bar{t})$, then a deduction $\mathcal{D}(\Gamma; r(\bar{t}) \in t(\bar{t}))$ satisfies $\phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ satisfies $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$.

If not $\vdash_N r(\bar{t}) \in t(\bar{t})$, then $\mathcal{D}$ satisfies $\phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ is $\mathcal{C}$-normalizable.

2.3.1 If $\vdash_N r(\bar{t}) \in t(\bar{t})$, then

$\phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$, which is a negation closed W-predicate by IH.

If not $\vdash_N r(\bar{t}) \in t(\bar{t})$, we must check that the predicate " $\mathcal{D}$ is a $\mathcal{C}$-normalizable deduction of $r(\bar{t}) \in t(\bar{t})$ " is a negation closed W-predicate. Clearly this predicate is a W-predicate. So assume

$$\left.\begin{array}{c} \neg\, r(\bar{t}) \in t(\bar{t}) \\ \vdots \\ \dfrac{\perp}{r(\bar{t}) \in t(\bar{t})} \end{array}\right\} \mathcal{D}$$

has the $\perp\left(\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}_{\bar{t}})\right)$ -property. Since not $\vdash_N r(\bar{t})\in t(\bar{t})$ there will be no contraction defined at the conclusion of $\mathcal{D}$ so $\mathcal{D}$ must be $\mathcal{C}$-normalizable which is what we want.

2.3.2 Assume $\alpha_{t_i} = \alpha'_{t_i}(\mathcal{D}_{s,u}, t_i(x))$, $i\leq n$.

We want to make sure that

$$\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}'_{\bar{t}})\left(\mathcal{D}_{s,u}, r(\bar{t}(x))\in t(\bar{t}(x))\right).$$

There will be two main cases depending on the form of $t(\bar{t}(x))$:

2.3.2.1 $t(\bar{t}(x))\equiv x$.

Here we will consider two subcases depending on whether x is free in $r(\bar{t}(x))$ or not:

2.3.2.1.1 x is free in $r(\bar{t}(x))$:

Assume we have a deduction $\mathcal{D}(\Gamma; r(\bar{t}(s))\in t(\bar{t}(s)))$ which satisfies $\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

So given $\dfrac{\mathcal{D} \quad \mathcal{D}_{s,u}}{r(\bar{t}(u))\in t(\bar{t}(u))}$ we have to make sure that this deduction satisfies $\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}'_{\bar{t}})$.

Now if not $\vdash_N r(\bar{t}(s))\in t(\bar{t}(s))$ this follows directly since both $\mathcal{D}$ and $\mathcal{D}_{s,u}$ are $\mathcal{C}$-normalizable. $\mathcal{D}_{s,u}$ will thus be a $\mathcal{C}$-normalizable major deduction along the $\mathcal{C}$-main branch and $\mathcal{D}$ will reduce to a deduction $\mathcal{D}'$ such that we will have no $\mathcal{C}$-main cut in $\dfrac{\mathcal{D}' \quad \mathcal{D}_{s,u}}{r(\bar{t}(u))\in t(\bar{t}(u))}$ and minor and major deductions along the $\mathcal{C}$-main branch will be $\mathcal{C}$-normalizable. And if there is a $\mathcal{C}$-main deduction this will also be $\mathcal{C}$-normalizable. So the deduction we started with is consequently $\mathcal{C}$-normaliz-

able and thus satisfies $\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$.

The other direction follows by using the same argument with obvious changes. If $\vdash_N r(\bar{t}(s)) \in t(\bar{t}(s))$ on the other hand we have to make sure that the contraction

$$\left(r(\bar{t}(x))\in u\right)\ \frac{\mathcal{D}''\left\{\begin{array}{l}\vdots\\ r(\bar{t}(s))\in s\\ \vdots\\ r(\bar{t}(s))\in u\end{array}\right.\ \begin{array}{l}\mathcal{D}'\\ \mathcal{D}_{s,u}\end{array}}{r(\bar{t}(u))\in u}$$

satisfies $\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$.

Since $\alpha_s = \alpha_u(\mathcal{D}_{s,u}, x)$ we know that $\mathcal{D}'$ satisfies $\alpha_u(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$, noting that $\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a negation closed W-object of type $r(\bar{t})$ by IH and that

$$\mathcal{D}''\left\{\begin{array}{l}\vdots\\ r(\bar{t}(s))\in s\end{array}\right.$$

satisfies $\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and so satisfies $\alpha_s(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$.

Now $\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{r(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{s,u}, r(\bar{t}(x)))$

by IH, and since α_u is a negation closed W-object of type u it follows that $\alpha_u(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}})) = \alpha_u(\phi_{r(\bar{x})}(\bar{\alpha}'_{\bar{t}'}))(\mathcal{D}_{s,u}, r(\bar{t}(x))\in u)$

which means the contraction satisfies $\alpha_u(\phi_{r(\bar{x})}(\bar{\alpha}'_{\bar{t}'}))$

that is $\phi_{r(\bar{x})\in t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$, which is what we want.

The other direction follows in the same manner.

2.3.2.1.2 x is not free in

If not $\vdash_N r(\bar{t}) \in S$ we want to make sure that $\dfrac{\mathcal{D} \quad \mathcal{D}_{S,u}}{r(\bar{t}) \in u}$

is $\mathcal{C}$-normalizable when $\mathcal{D}$ satisfies $\phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}_{\bar{t}})$

That means we have to show that the contraction

$$\left.\begin{array}{c} \vdots \\ r(\bar{t}) \in S \\ \vdots \\ r(\bar{t}) \in u \end{array}\right\} \mathcal{D}'$$

is $\mathcal{C}$-normalizable. We know that

$\vdots$
$r(\bar{t}) \in S$ is $\mathcal{C}$-normalizable and we also know that

$r(\bar{t}) \in S$
$\vdots$
$r(\bar{t}) \in u$ is $\mathcal{C}$-normalizable, this being a condition on $\mathcal{D}_{S,u}$

To show that $\mathcal{D}'$ is $\mathcal{C}$-normalizable we show the following:

If not $\vdash_N r \in t$ and

$$\left.\begin{array}{c} r \in t \\ \vdots \\ A \end{array}\right\} \mathcal{D}_A$$

is $\mathcal{C}$-normalizable, then if

$\vdots$
$r \in t$ is $\mathcal{C}$-normalizable, $\quad \begin{array}{c} \vdots \\ r \in t \\ \vdots \\ A \end{array}$

will be $\mathcal{C}$-normalizable.

This will be shown by main induction on the definition of "$\mathcal{D}_A$ is a $\mathcal{C}$-normalizable".

If $\mathcal{D}_A$ is just A the claim is trivially true.

If $\mathcal{D}_A$ is on I-form the desired result follows directly from IH.

If $\mathcal{D}_A$ is on E-form and has a $\mathcal{C}$-main cut we make the contraction at this cut and use IH to get the desired conclusion.

If $\mathcal{D}_A$ is on E-form and has no $\mathcal{C}$-main cut we have to consider two cases:

(i) The assumption $r \in t$ is at the top of a minor or major deduction along the $\mathcal{C}$-main branch or at the top of the $\mathcal{C}$-main deduction. In this case we only have to apply IH.

(ii) The assumption $r \in t$ is at the top of the $\mathcal{C}$-main branch of $\mathcal{D}_A$.
We will then prove the result by induction on the definition of

$$\left.\begin{array}{c} \vdots \\ r \in t \end{array}\right\} \mathcal{D}_{r \in t} \text{ is } \mathcal{C}\text{-normalizable".}$$

If $\mathcal{D}_{r \in t}$ is just $r \in t$ we are done.
If $\mathcal{D}_{r \in t}$ is on I-form this deduction will then be the $\mathcal{C}$-main deduction in

$$\begin{array}{c} \vdots \\ r \in t \\ \vdots \\ A \end{array}$$

since there will be no rule of contraction defined at the cut $r \in t$.

Furthermore $\mathcal{D}_{r\epsilon\bar{t}}$ is $\mathcal{C}$-normalizable, so we have what we want. If $\mathcal{D}_{r\epsilon\bar{t}}$ is on E-form and has a $\mathcal{C}$-main cut the desired result follows directly by using IH and if $\mathcal{D}_{r\epsilon\bar{t}}$ is on E-form and has no $\mathcal{C}$-main cut we are immediately done. To conclude we thus know

$$\left.\begin{array}{c} \vdots \\ r(\bar{t})\epsilon S \\ \vdots \\ r(\bar{t})\epsilon u \end{array}\right\} \mathcal{D}'$$

that is $\mathcal{C}$-normalizable. We use the same argument with obvious chances in the other direction.

If $\vdash_N r(\bar{t})\epsilon S$ we know that $\mathcal{D}$ satisfies $\alpha_S(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$.

Now since $\alpha_S = \alpha_u(\mathcal{D}_{S,u}, x)$ it follows that the contraction

$$\begin{array}{c} \vdots \\ r(\bar{t})\epsilon S \\ \vdots \\ r(\bar{t})\epsilon u \end{array}$$

satisfies $\alpha_u(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$,

that is $\phi_{r(\bar{x})\epsilon\bar{t}(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$, which is what we want. As in previous cases, to show the result in the other direction we use the same argument with trivial changes.

2.3.2.2 $t(\bar{t}(x)) \not\equiv x$.

The trivial case when x is not free in $r(\bar{t}(x)) \in t(\bar{t}(x))$ follows directly.

If not $\vdash_N r(\bar{t}(s)) \in t(\bar{t}(s))$ we reason as in 2.3.2.1.1, so assume $\vdash_N r(\bar{t}(s)) \in t(\bar{t}(s))$:

By IH we know that $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{s,u}, t(\bar{t}(x)))$

and $\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{r(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{s,u}, r(\bar{t}(x)))$,

so then it follows that

$$\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}})) = \phi_{t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\phi_{r(\bar{x})}(\bar{\alpha}'_{\bar{t}'}))(\mathcal{D}_{s,u}, r(\bar{t}(x)) \in t(\bar{t}(x))).$$

So clearly we have

$$\phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{r(\bar{x}) \in t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{s,u}, r(\bar{t}(x)) \in t(\bar{t}(x))).$$

2.4 $T(\bar{x}) \equiv r(\bar{x}) = t(\bar{x})$.

By IH $\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ are negation closed W-objects of type $r(\bar{t})$ and $t(\bar{t})$ respectively. So given a negation closed W-object α_u of type u we know by 2.3 that $\phi_{x \in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and $\phi_{x \in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ are negation closed W-predicates of type $u \in r(\bar{t})$ and $u \in t(\bar{t})$ respectively. Using this we define $\phi_{r(\bar{x}) = t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ in the following way:

$r(\bar{t}) = t(\bar{t})$ satisfies $\phi_{r(\bar{x}) = t(\bar{x})}(\bar{\alpha}_{\bar{t}})$

If $\mathcal{D}(\top ; r(\bar{t})=t(\bar{t}))$ is on I-form

$$\frac{\begin{array}{cc} \cancel{x\in r(\bar{t})} & \cancel{x\in t(\bar{t})} \\ \vdots & \vdots \\ x\in t(\bar{t}) & x\in r(\bar{t}) \end{array}}{r(\bar{t})=t(\bar{t})}$$

then $\mathcal{D}$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff for all terms u and

all negation closed W-objects α_u of type u

$$\begin{array}{cc} \vdots & \vdots \\ u\in r(\bar{t}) & u\in t(\bar{t}) \\ \vdots & \vdots \\ u\in t(\bar{t}) & u\in r(\bar{t}) \end{array}$$

satisfies $\phi_{x\in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and $\phi_{x\in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$

$\vdots$ $\vdots$

respectively for all $u\in r(\bar{t})$, $u\in t(\bar{t})$ satisfying

$\phi_{x\in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and $\phi_{x\in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ respectively.

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$

iff $\mathcal{D}$ has the $E(\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property.

2.4.1 $r(\bar{t})=t(\bar{t})$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and if $\mathcal{D}$ is on

E-form $\mathcal{D}$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ has the

$E(\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property, so to make sure that

$\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a W-predicate of type $r(\bar{t})=t(\bar{t})$

we have to convince ourselves that if $\mathcal{D}$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$, then $\mathcal{D}$ is $\mathcal{C}$-normalizable. Now it is easily seen that this amounts to checking that $\mathcal{D}$ is $\mathcal{C}$-normalizable when $\mathcal{D}$ is on I-form. What we know is that for all terms u and all negation closed W-objects α_u of type u

$$\begin{array}{cc} \vdots & \vdots \\ u \in r(\bar{t}) & u \in t(\bar{t}) \\ \vdots & \vdots \\ u \in t(\bar{t}) & u \in r(\bar{t}) \end{array}$$

satisfies $\phi_{x \in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and $\phi_{x \in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$

$$\vdots \qquad \vdots$$

respectively when $u \in r(\bar{t})$, $u \in t(\bar{t})$ satisfies $\phi_{x \in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and $\phi_{x \in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ respectively.

Now x has no E_N predecessors so $\alpha_x \underset{def.}{=} x$ will be a negation closed W-object and clearly $x \in r(\bar{t})$ and $x \in t(\bar{t})$ satisfy $\phi_{x \in r(\bar{x})}(\alpha_x, \bar{\alpha}_{\bar{t}})$ and $\phi_{x \in t(\bar{x})}(\alpha_x, \bar{\alpha}_{\bar{t}})$ respectively,

so by IH and 2.3 it follows that

$$\begin{array}{cc} x \in r(\bar{t}) & x \in t(\bar{t}) \\ \vdots & \vdots \\ x \in t(\bar{t}) & x \in r(\bar{t}) \end{array}$$

are $\mathcal{C}$-normalizable and we are done.

To see that $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ also is negation closed assume

$$\left.\begin{array}{c} \neg\, r(\bar{t}) = t(\bar{t}) \\ \vdots \\ \dfrac{\bot}{r(\bar{t}) = t(\bar{t})} \end{array}\right\} \mathcal{D}$$

has the $\bot\left(\phi_{r(\bar{x})=t(\bar{x})}{}^{(\bar{\alpha}_{\bar{t}})}\right)$ -property.

To show that $\mathcal{D}$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}{}^{(\bar{\alpha}_{\bar{t}})}$ we have to show that the contraction

$$\dfrac{\dfrac{\dfrac{x \in r(\bar{t}) \quad r(\bar{t}) = t(\bar{t})}{x \in t(\bar{t})} \quad \neg\, x \in t(\bar{t})}{\bot}}{\neg\, r(\bar{t}) = t(\bar{t})} \qquad \dfrac{\dfrac{\dfrac{x \in t(\bar{t}) \quad r(\bar{t}) = t(\bar{t})}{x \in r(\bar{t})} \quad \neg\, x \in r(\bar{t})}{\bot}}{\neg\, r(\bar{t}) = t(\bar{t})}$$

$$\vdots \qquad\qquad \vdots$$

$$\dfrac{\bot}{x \in t(\bar{t})} \qquad\qquad \dfrac{\bot}{x \in r(\bar{t})}$$

$$\dfrac{x \in t(\bar{t}) \qquad\qquad x \in r(\bar{t})}{r(\bar{t}) = t(\bar{t})}$$

satisfies $\phi_{r(\bar{x})=t(\bar{x})}{}^{(\bar{\alpha}_{\bar{t}})}$.

So given a term u, a negation closed W-object α_u of type u and a deduction

$$\vdots$$

$$u \in r(\bar{t}) \text{ satisfying } \phi_{x \in r(\bar{x})}{}^{(\alpha_u, \bar{\alpha}_{\bar{t}})}$$

we have to show that

$$\mathcal{D}_2\left\{\begin{array}{c}\vdots\\ \dfrac{u\in r(\bar{t})\quad r(\bar{t})=t(\bar{t})}{u\in t(\bar{t})}\quad \dfrac{}{\neg u\in t(\bar{t})}\\ \dfrac{\bot}{\neg\, r(\bar{t})=t(\bar{t})}\\ \vdots\\ \dfrac{\bot}{u\in t(\bar{t})}\end{array}\right\}\mathcal{D}_1$$

satisfies $\phi_{x\in t(\bar{x})}(\alpha_u,\bar{\alpha}_{\bar{t}})$, and similarly for the other premise deduction. By IH and 2.3 we know that $\phi_{x\in t(\bar{x})}(\alpha_u,\bar{\alpha}_{\bar{t}})$ is a negation closed W-predicate of type $u\in t(\bar{t})$, so it suffices to show that $\mathcal{D}_1$ has the $\bot(\phi_{x\in t(\bar{x})}(\alpha_u,\bar{\alpha}_{\bar{t}}))$-property.

So first of all we have to make sure that $\mathcal{D}_2$ is $\mathcal{C}$-normalizable. We have assumed that $\mathcal{D}$ has the $\bot(\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}}))$-property, so this follows if

$$\mathcal{D}_4\left\{\begin{array}{c}\dfrac{\begin{array}{cc}\vdots & \vdots\\ u\in r(\bar{t}) & r(\bar{t})=t(\bar{t})\end{array}}{u\in t(\bar{t})}\quad \dfrac{}{\neg u\in t(\bar{t})}\\ \dfrac{\bot}{\vdots}\end{array}\right\}\mathcal{D}_3$$

is $\mathcal{C}$-normalizable for all $r(\bar{t})=t(\bar{t})$ satisfying $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$. This amounts to showing that

$$\mathcal{D}_4\left\{\begin{array}{cc} \vdots & \vdots \\ u\in r(\bar{t}) & r(\bar{t})=t(\bar{t}) \end{array}\right\}\mathcal{D}_3$$
$$u\in t(\bar{t})$$

is $\mathcal{C}$-normalizable. In order to prove this we will prove that if $\mathcal{D}_3$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$

and $\mathcal{D}_4$ satisfies $\phi_{x\in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$, then

$$\mathcal{D}_4\left\{\begin{array}{cc} \vdots & \vdots \\ u\in r(\bar{t}) & r(\bar{t})=t(\bar{t}) \end{array}\right\}\mathcal{D}_3$$
$$u\in t(\bar{t})$$

satisfies $\phi_{x\in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$.

In doing this I will use a property that will be used frequently in what follows:
Let P be a predicate defined for deductions. A deduction $\mathcal{D}$ is said to have the Red(P)-property of either $\mathcal{D}$ is or reduces to a deduction $\mathcal{D}'$ consisting only of an assumption
or
$\mathcal{D}$ is or reduces to a deduction $\mathcal{D}'$ which is on I-form and satisfies P
or
$\mathcal{D}$ is or reduces to a deduction $\mathcal{D}'$ on E-form such that $\mathcal{D}'$ has no $\mathcal{C}$-main cut, the minor and major deductions along the $\mathcal{C}$-main branch of $\mathcal{D}'$ are $\mathcal{C}$-normalizable, and the $\mathcal{C}$-main deduction, if there is one, is $\mathcal{C}$-normalizable.
Now if α_A is a W-predicate and $\mathcal{D}(\Gamma; A)$ satisfies α_A , then $\mathcal{D}$ has the Red(α_A)-property:

If $\mathcal{D}$ is just an assumption or on I-form then this is trivial. So assume $\mathcal{D}$ is on E-form. If $\mathcal{D}$ has no $\mathcal{C}$-main cut the

conclusion immediately follows.
If $\mathcal{D}$ has a $\mathcal{C}$-main cut we know that $\mathcal{D}$ must eventually reduce to a deduction $\mathcal{D}'$ which is either an assumption, on I-form satisfying α_A or to a deduction having no $\mathcal{C}$-main cut and still satisfying α_A and so satisfying the other conditions.

By assumption $\mathcal{D}_3$ above satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\mathcal{D}_4$ satisfies $\phi_{x\in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and so $\mathcal{D}_3$ has the $\mathrm{Red}(\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}}))$-property.
Using this we have two cases to consider:
$\mathcal{D}_3$ eventually reduces to a deduction on E-form having no $\mathcal{C}$-main cut. Since $\mathcal{D}_4$ satisfies $\phi_{x\in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ we will have a deduction on E-form having no $\mathcal{C}$-main cut, $\mathcal{C}$-normalizable minor and major deductions along the $\mathcal{C}$-main branch and if the deduction has a $\mathcal{C}$-main deduction this is $\mathcal{C}$-normalizable. So we are done.
If $\mathcal{D}_3$ on the other hand reduces to a deduction on I-form we have to make sure the contraction of

$$\dfrac{\mathcal{D}_4\left\{\begin{array}{c}\vdots\\ u\in r(\bar{t})\end{array}\right. \qquad \left.\begin{array}{c} \cancel{x\in r(\bar{t})} \quad \cancel{x\in t(\bar{t})} \\ \vdots \qquad\quad \vdots \\ x\in t(\bar{t}) \quad x\in r(\bar{t}) \\ \hline r(\bar{t})=t(\bar{t}) \end{array}\right\}\mathcal{D}_3'}{u\in t(\bar{t})}$$

satisfies $\phi_{x\in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$. But since $\mathcal{D}_3'$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\mathcal{D}_4$ satisfies $\phi_{x\in r(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ this follows directly.

So $\dfrac{\mathcal{D}_4 \quad \mathcal{D}_3}{u \in t(\bar{t})}$ satisfies $\phi_{x \in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ and

is thus $\mathcal{C}$-normalizable.

Now given a deduction

$$\dfrac{\mathcal{D}_5 \left\{ \begin{array}{c} \not{u \in t(\bar{t})} \\ \vdots \\ \bot \end{array} \right.}{\neg u \in t(\bar{t})}$$

such that

$$\begin{array}{c} \vdots \\ u \in t(\bar{t}) \\ \vdots \\ \bot \\ \vdots \\ u \in t(\bar{t}) \end{array}$$

is $\mathcal{C}$-normalizable for all

satisfying $\phi_{x \in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$.

We want to see that

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ u \in r(\bar{t}) \quad \not{r(\bar{t}) = t(\bar{t})} \end{array} \quad \left. \begin{array}{c} \vdots \\ \not{u \in t(\bar{t})} \end{array} \right\} \mathcal{D}_5 \quad \neg u \in t(\bar{t})}{\bot}}{\neg r(\bar{t}) = t(\bar{t})}$$

$$\begin{array}{c} \vdots \\ \bot \end{array}$$

is $\mathcal{C}$-normalizable.

Now by assumption $\mathcal{D}$ has the $\bot(\phi_{r(\bar{x}) = t(\bar{x})}(\bar{\alpha}_{\bar{t}}))$-

property, so it will be enough to show that

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ u\in r(\bar{t}) & r(\bar{t})=t(\bar{t})\end{array}\Big\}\mathcal{D}_6}{u\in t(\bar{t})}\quad \Big\}\mathcal{D}_5}{\neg u\in t(\bar{t})}$$
$$\bot$$

is $\mathcal{C}$-normalizable if $\mathcal{D}_6$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

To establish this we have to show that the contraction

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ u\in r(\bar{t}) & r(\bar{t})=t(\bar{t})\end{array}}{u\in t(\bar{t})}$$
$$\vdots$$
$$\bot \quad \Big\}\mathcal{D}_5$$

is $\mathcal{C}$-normalizable.

By the choice of $\mathcal{D}_5$ this follows if

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ u\in r(\bar{t}) & r(\bar{t})=t(\bar{t})\end{array}}{u\in t(\bar{t})}$$

satisfies $\phi_{x\in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$

which it does as we have seen by previous reasoning.
So $\mathcal{D}_1$ has the $\bot\,(\phi_{x\in t(\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}}))$ -property.

The argument for the other premise deduction of our contraction is with obvious changes the same as the one given here.

2.4.2 Assume $\alpha_{t_i} = \alpha'_{t_i}\left(\mathcal{D}_{s,u}, t_i(x)\right)$, $i \leq n$.

We want to see that

$$\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}'_{\bar{t}},)\left(\mathcal{D}_{s,u}, r(\bar{t}(x))=t(\bar{t}(x))\right).$$

So assume the deduction $\mathcal{D}\left(\Gamma; r(\bar{t}(s))=t(\bar{t}(s))\right)$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

To show that $\dfrac{\mathcal{D} \quad \mathcal{D}_{s,u}}{r(\bar{t}(u))=t(\bar{t}(u))}$ satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}'_{\bar{t}},)$

we have to show that the contraction

$$\dfrac{\dfrac{\dfrac{\cancel{x\in r(\bar{t}(u))} \quad \mathcal{D}_{s,u}}{x\in r(\bar{t}(s)) \quad \mathcal{D}}}{\dfrac{x\in t(\bar{t}(s)) \quad \mathcal{D}_{s,u}}{x\in t(\bar{t}(u))}} \qquad \dfrac{\dfrac{\cancel{x\in t(\bar{t}(u))} \quad \mathcal{D}_{s,u}}{x\in t(\bar{t}(s)) \quad \mathcal{D}}}{\dfrac{x\in r(\bar{t}(s)) \quad \mathcal{D}_{s,u}}{x\in r(\bar{t}(u))}}}{r(\bar{t}(u))=t(\bar{t}(u))}$$

satisfies $\phi_{r(\bar{x})=t(\bar{x})}(\bar{\alpha}'_{\bar{t}},)$. To show this we have to show that for any term v and any negation closed W-object α_v of type v, if $\mathcal{D}_v\left(\Gamma; v\in t(\bar{t}(u))\right)$ satisfies $\phi_{x\in t(\bar{x})}(\alpha_v, \bar{\alpha}'_{\bar{t}},)$, then

$$\dfrac{\dfrac{\mathcal{D}_v \quad \mathcal{D}_{s,u}}{v\in t(\bar{t}(s)) \quad \mathcal{D}}}{v\in r(\bar{t}(s)) \quad \mathcal{D}_{s,u}}{v\in r(\bar{t}(u))}$$

satisfies $\phi_{x\in r(\bar{x})}(\alpha_v, \bar{\alpha}'_{\bar{t}},)$ and similarly for the other premise deduction.

First we show that $\dfrac{\mathcal{D}_v \quad \mathcal{D}_{s,u}}{v \in t(\bar{t}(s))}$ satisfies

$\phi_{x \in t(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$. If $\vdash_N v \in t(\bar{t}(s))$, then we use IH to see that $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\mathcal{D}_{s,u}, t(\bar{t}(x)))$.

Since α_v is a negation closed W-object $\alpha_v = \alpha_v(\mathcal{D}_{s,u}, v)$

so then we have $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})(\alpha_v) = \phi_{t(\bar{x})}(\bar{\alpha}'_{\bar{t}'})(\alpha_v)(\mathcal{D}_{s,u}, v \in t(\bar{t}(x)))$.

If not $\vdash_N v \in t(\bar{t}(s))$ we reason as in 2.3.2.1.1.

Now $\mathcal{D}$ satisfies $\phi_{r(\bar{x}) = t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and so has the

$Red(\phi_{r(\bar{x}) = t(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property. Reasoning as in 2.4.1 we see then that

$$\frac{\dfrac{\mathcal{D}_v \quad \mathcal{D}_{s,u}}{v \in t(\bar{t}(s)) \quad \mathcal{D}}}{v \in r(\bar{t}(s))}$$

satisfies $\phi_{x \in r(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$.

So by reasoning as above we see that

$$\frac{\dfrac{\dfrac{\mathcal{D}_v \quad \mathcal{D}_{s,u}}{v \in t(\bar{t}(s)) \quad \mathcal{D}}}{v \in r(\bar{t}(s)) \quad \mathcal{D}_{s,u}}}{v \in r(\bar{t}(u))}$$

satisfies $\phi_{x \in r(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$.

The same type of argument, with obvious changes, gives the desired result for the other premise deduction.
We reason in the same manner to prove the other direction.

2.5 $T(\bar{x}) \equiv A(\bar{x}) \& B(\bar{x})$.

By IH $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ are negation closed W-predicates of type $A(\bar{t})$ and $B(\bar{t})$ respectively. Using this we define $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ as follows:

$A(\bar{t}) \& B(\bar{t})$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}$ is on I-form

$$\frac{\begin{array}{cc} \vdots & \vdots \\ A(\bar{t}) & B(\bar{t}) \end{array}}{A(\bar{t}) \& B(\bar{t})}$$

$\mathcal{D}$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff

$\vdots$ $\qquad\qquad$ $\vdots$

$A(\bar{t})$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $B(\bar{t})$ satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}(\Gamma\,;\, A(\bar{t}) \& B(\bar{t}))$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ has the $E(\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}}))$-
-property.

2.5.1 It is easily seen that $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a W-predicate of type $A(\bar{t}) \& B(\bar{t})$ observing that if $\mathcal{D}$ is on I-form and satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ the premise deductions of the last inference in $\mathcal{D}$ satisfy $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and

$\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ respectively and thus by IH are $\mathcal{C}$-

normalizable.

Now assume the deduction

$$\left.\begin{array}{c} \neg(A\&B) \\ \vdots \\ \dfrac{\bot}{A\&B} \end{array}\right\} \mathcal{D}$$

has the $\bot(\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property.

To see that $\mathcal{D}$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ we have to convince ourselves that the contraction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\&B}{A} \quad \neg A}{\bot}}{\neg(A\&B)}}{\vdots}}{\dfrac{\bot}{A}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\&B}{B} \quad \neg B}{\bot}}{\neg(A\&B)}}{\vdots}}{\dfrac{\bot}{B}}$$

$$A\&B$$

satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

That this is so is not hard to see using IH following the same line of thought as in 2.4.1.

Thus $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a W-predicate closed under negation.

2.5.2 We assume $\alpha_{t_i} = \alpha'_{t_i}(\mathcal{D}_{s,u}, t_i(x))$, $i \leq n$.

to show $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, A(\bar{t}(x))\&B(\bar{t}(x)))$.

So assume we have a deduction $\mathcal{D}(\Gamma; A(\bar{t}(s))\&B(\bar{t}(s)))$ satisfying $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

Now $\dfrac{\mathcal{D} \quad \mathcal{D}_{s,u}}{A(\bar{t}(u))\&B(\bar{t}(u))}$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}'_{\bar{t}})$ if the contraction

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{A(\bar{t}(s))} \quad \mathcal{D}_{s,u}}{A(\bar{t}(u))} \quad \dfrac{\dfrac{\mathcal{D}}{B(\bar{t}(s))} \quad \mathcal{D}_{s,u}}{B(\bar{t}(u))}}{A(\bar{t}(u))\&B(\bar{t}(u))}$$

does. So take one of the premise deductions

$$\left.\dfrac{\dfrac{\mathcal{D}}{A(\bar{t}(s))} \quad \mathcal{D}_{s,u}}{A(\bar{t}(u))}\right\} \mathcal{D}_1$$

By IH $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{A(\bar{x})}(\bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, A(\bar{t}(x)))$

so to see that $\mathcal{D}_1$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}'_{\bar{t}})$ it is enough to make sure that $\dfrac{\mathcal{D}}{A(\bar{t}(s))}$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

Now $\mathcal{D}$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and so has the

$Red(\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property.

It is easily seen that the only case we have to check is when $\mathcal{D}$ reduces to I-form:

$$\left.\begin{array}{c} \vdots \qquad\quad \vdots \\ \dfrac{A(\bar{t}(s)) \qquad B(\bar{t}(s))}{A(\bar{t}(s)) \,\&\, B(\bar{t}(s))} \\ A(\bar{t}(s)) \end{array}\right\} \mathcal{D}_2$$

$\mathcal{D}_2$ satisfies $\phi_{A(\bar{x})\,\&\,B(\bar{x})}(\bar{\alpha}_{\bar{t}})$, so

$\vdots$

$A(\bar{t}(s))$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and then so does the contraction,

so $\dfrac{\mathcal{D}}{A(\bar{t}(s))}$ does indeed satisfy $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

We use the same argument with obvious changes in the other direction.

2.6 $T(\bar{x}) \equiv A(\bar{x}) \to B(\bar{x})$.

By IH $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ are negation closed W-objects of type $A(\bar{t})$ and $B(\bar{t})$. Using this we define $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ as follows:

$A(\bar{t}) \to B(\bar{t})$ satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}(\Gamma\,;\, A(\bar{t}) \to B(\bar{t}))$ is on I-form

$$\begin{array}{c} \cancel{A(\bar{t})} \\ \vdots \\ B(\bar{t}) \\ \hline A(\bar{t}) \to B(\bar{t}) \end{array}$$

$\mathcal{D}$ satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff

$A(\bar{t})$

$\vdots$

$B(\bar{t})$

satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ for all $A(\bar{t})$ satisfying $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ has the $E(\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property.

2.6.1 To see that $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a W-predicate of type $A(\bar{t})\to B(\bar{t})$ we observe that if $\mathcal{D}$ is on I-form

$$\frac{\begin{array}{c}\not{A(\bar{t})}\\ \vdots \\ B(\bar{t})\end{array}}{A(\bar{t})\to B(\bar{t})}$$

and satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$, then

$A(\bar{t})$

$\vdots$

$B(\bar{t})$

will satisfy $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ since by IH $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a W-predicate of type $A(\bar{t})$, and so $A(\bar{t})$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$. So by IH

$A(\bar{t})$

$\vdots$

$B(\bar{t})$ is $\mathcal{C}$-normalizable, which means $\mathcal{D}$ is $\mathcal{C}$-normalizable.

So assume

$$\left.\begin{array}{c} \neg(A \to B) \\ \vdots \\ \bot \\ \hline A \to B \end{array}\right\} \mathcal{D}$$

is a deduction having the $\bot\left(\phi_{A(\bar{x}) \to B(\bar{x})}(\bar{\alpha}_{\bar{t}})\right)$ -property, To see that $\mathcal{D}$ satisfies $\phi_{A(\bar{x}) \to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ we have to check that the contraction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \quad A \to B}{B} \quad \neg B}{\bot}}{\neg(A \to B)} \\ \vdots \\ \dfrac{\bot}{B}}}{A \to B}}$$

satisfies $\phi_{A(\bar{x}) \to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

By IH $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a negation closed W-predicate, so as in 2.4.1 it sufficies to make sure that for any $\begin{array}{c}\vdots \\ A\end{array}$ satisfying $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A} \quad A \to B}{B} \quad \neg B}{\bot}}{\neg(A \to B)} \\ \vdots \\ \dfrac{\bot}{B}}$$

has the $\bot\left(\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})\right)$ -property.

This follows using the same type of argument used in 2.4.1.

2.6.2 Assume $\alpha_{t_i} = \alpha'_{t_i}(\mathcal{D}_{s,u}, t_i(x))$.

To show $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, A(\bar{t}(x))\to B(\bar{t}(x)))$

we note first of all that by IH $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{A(\bar{x})}(\bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, A(\bar{t}(x)))$

and $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{B(\bar{x})}(\bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, B(\bar{t}(x)))$.

So given a deduction $\mathcal{D}(\Gamma;\ A(\bar{t}(s))\to B(\bar{t}(s)))$ satisfying $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$, to see that $\dfrac{\mathcal{D} \quad \mathcal{D}_{s,u}}{A(\bar{t}(u))\to B(\bar{t}(u))}$ satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}'_{\bar{t}})$ we have to make sure that the contraction

$$\dfrac{\dfrac{\dfrac{\dfrac{A(\bar{t}(u)) \quad \mathcal{D}_{s,u}}{A(\bar{t}(s)) \quad \mathcal{D}}}{B(\bar{t}(s)) \quad \mathcal{D}_{s,u}}}{B(\bar{t}(u))}}{A(\bar{t}(u))\to B(\bar{t}(u))}$$

satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}'_{\bar{t}})$, which means we have to see that

$$\left.\begin{array}{l}\vdots\\ \dfrac{\dfrac{\dfrac{A(\bar{t}(u)) \quad \mathcal{D}_{s,u}}{A(\bar{t}(s)) \quad \mathcal{D}}}{B(\bar{t}(s)) \quad \mathcal{D}_{s,u}}}{B(\bar{t}(u))}\end{array}\right\}\ \mathcal{D}_1$$

satisfies $\phi_{B(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$ for all

$\vdots$

$A(\bar{t}(u))$

satisfying $\phi_{A(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$.

By IH

$$\left.\begin{array}{c}\vdots \\ \dfrac{A(\bar{t}(u)) \quad \mathcal{D}_{s,u}}{A(\bar{t}(s))}\end{array}\right\} \mathcal{D}_2$$

then satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$. Since $\mathcal{D}$ satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ it has the $Red(\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property, so as in previous cases we only have to pay special attention to the situation when $\mathcal{D}$ reduces to I-form:

$$\mathcal{D}_2 \left\{\begin{array}{c} \dfrac{\begin{array}{cc}\vdots & \cancel{A(\bar{t}(s))} \\ \vdots & \vdots \\ A(\bar{t}(u)) \ \mathcal{D}_{s,u} & B(\bar{t}(s))\end{array}}{\begin{array}{cc} A(\bar{t}(s)) & A(\bar{t}(s))\to B(\bar{t}(s))\end{array}} \\ B(\bar{t}(s)) \end{array}\right\} \mathcal{D}'$$

But then clearly the contraction satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ since $\mathcal{D}'$ satisfies $\phi_{A(\bar{x})\to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and by a previous remark $\mathcal{D}_2$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$. So by IH then it follows that $\mathcal{D}_1$ satisfies $\phi_{B(\bar{x})}(\bar{\alpha}'_{\bar{t}'})$ which is what we want.

2.7 $T(\bar{x}) \equiv \forall x A(x,\bar{x})$.

By IH for any u and any negation closed W-object α_u of type u

$\phi_{A(x,\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$ is a negation closed W- predicate of type $A(u,\bar{t})$. Using this we define $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ as follows:

$\forall x A(x,\bar{t})$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$

If $\mathcal{D}(\Gamma\,;\ \forall x A(x,\bar{t}))$ is on I-form

$$\vdots$$

$$\frac{A(x,\bar{t})}{\forall x\, A(x,\bar{t})}$$

$\mathcal{D}$ satisfies $\phi_{\forall x\, A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff for all u and all negation closed W-objects α_u of type u

$$\vdots$$

$A(u,\bar{t})$

satisfies $\phi_{A(x,\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{\forall x\, A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\mathcal{D}$ has the $E\left(\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})\right)$-property.

2.7.1 To see that $\phi_{\forall x\, A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ is a W-predicate note that if $\mathcal{D}$ is on I-form

$$\vdots$$

$$\frac{A(x,\bar{t})}{\forall x A(x,\bar{t})}$$

and satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$, then

$$\left.\begin{array}{c}\vdots\\ A(x,\bar{t})\end{array}\right\}\mathcal{D}'$$

satisfies $\phi_{A(x,\bar{x})}(\alpha_x,\bar{\alpha}_{\bar{t}})$, since indeed there exist negation closed W-objects α_x of type x, so by IH $\mathcal{D}'$ is $\mathcal{C}$-normalizable, and thus so is $\mathcal{D}$.

So assume we have a deduction

$$\left.\begin{array}{c}\cancel{\neg\forall x A(x)}\\ \vdots\\ \dfrac{\bot}{\forall x A(x)}\end{array}\right\}\mathcal{D}$$

such that $\mathcal{D}$ has the $\bot(\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property.

To see that $\mathcal{D}$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ we have to make sure that the contraction

$$\left.\begin{array}{c}\dfrac{\dfrac{\cancel{\forall x A(x)}}{A(x)}\quad\cancel{\neg A(x)}}{\dfrac{\bot}{\neg\forall x A(x)}}\\ \vdots\\ \dfrac{\dfrac{\bot}{A(x)}}{\forall x A(x)}\end{array}\right\}\mathcal{D}_1$$

satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$.

So given a term u and a negation closed W-object α_u of type u we have to see that

$$\left.\begin{array}{l} \dfrac{\dfrac{\dfrac{\forall x A(x)}{A(u)} \quad \neg A(u)}{\bot}}{\neg \forall x A(x)} \\ \vdots \\ \dfrac{\bot}{A(u)} \end{array}\right\} \mathcal{D}_2$$

satisfies $\phi_{A(x,\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})$. By IH this follows if $\mathcal{D}_2$ has the $\bot\left(\phi_{A(x,\bar{x})}(\alpha_u, \bar{\alpha}_{\bar{t}})\right)$ -property. That this is so follows by reasoning similar to the argument given in 2.4.1.

2.7.2 Assume $\alpha_{t_i} = \alpha'_{t_i}(\mathcal{D}_{s,u}, t_i(z))$, $i \leq n$.

We will show $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{\forall x A(x,\bar{x})}(\bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, \forall x A(x, \bar{t}(z)))$.

By IH $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}}) = \phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}'_{\bar{t}})(\mathcal{D}_{s,u}, A(r, \bar{t}(z)))$

for all negation closed W-objects α_r.

This is so since for every negation closed W-object α_r

$\alpha_r = \alpha_r(\mathcal{D}_{s,u}, r)$.

So given a deduction $\mathcal{D}(\Gamma; \forall x A(x, \bar{t}(s)))$.

To see that $\dfrac{\mathcal{D} \quad \mathcal{D}_{s,u}}{\forall x\, A(x, \bar{t}(u))}$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}'_{\bar{t}})$

we have to see that the contraction

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{A(x,\bar{t}(s))}\quad \mathcal{D}_{s,u}}{A(x,\bar{t}(u))}}{\forall x\, A(x,\bar{t}(u))}$$

satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}'_{\bar{t}},)$, which means we have to make sure that for any r and any negation closed W-object α_r of type r

$$\dfrac{\dfrac{\mathcal{D}}{A(r,\bar{t}(s))}\quad \mathcal{D}_{s,u}}{A(r,\bar{t}(u))}$$

satisfies $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}'_{\bar{t}},)$. By IH it is then enough to make sure that $\dfrac{\mathcal{D}}{A(r,\bar{t}(s))}$ satisfies $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$.

Now since $\mathcal{D}$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ it has the $Red\left(\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})\right)$ -property.

As in previous cases we only have to check the situation when $\mathcal{D}$ reduces on I-form since in all other possible situations we are immediately done:

$$\left.\dfrac{\dfrac{\begin{array}{c}\vdots\\ A(x,\bar{t}(s))\end{array}}{\forall x A(x,\bar{t}(s))}}{A(r,\bar{t}(s))}\right\} \mathcal{D}'$$

That the contraction does indeed satisfy $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$

follows directly from the fact that $\mathcal{D}$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$

so $\dfrac{\mathcal{D}}{A(r,\bar{t}(s))}$ satisfies $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$, which is what we want.

2.8 $T(\bar{x}) \equiv \{x \mid A(x,\bar{x})\}$.

By IH for any r and any negation closed W-object α_r of type r

$\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$ is a negation closed W-predicate of type

$A(r,\bar{t})$.

Using this we define $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ as follows:

If $\{x \mid A(x,\bar{t})\}$ has no E_N predecessors

$$\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}}) \underset{def.}{=} \{x \mid A(x,\bar{t})\}.$$

If $\{x \mid A(x,\bar{t})\}$ has E_N predecessors $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$

is to be a function defined for negation closed W-objects α_r

of type r where $E_N(r, \{x \mid A(x,\bar{t})\})$ such that

$\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ is the following predicate defined

for deductions ending with $r \in \{x \mid A(x,\bar{t})\}$;

$r \in \{x \mid A(x,\bar{t})\}$ satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$.

If $\mathcal{D}(\Gamma;\ r \in \{x \mid A(x,\bar{t})\})$ is on I-form

$$\vdots$$

$$\frac{A(r,\bar{t})}{r \in \{x \mid A(x,\bar{t})\}}$$

then $\mathcal{D}$ satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ iff

$$\vdots$$

$A(r,\bar{t})$ satisfies $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ iff $\mathcal{D}$ has the $E\left(\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)\right)$ -property.

2.8.1 To see that $\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ is a negation closed W-object of type $\{x|A(x,\bar{t})\}$ we have to first make sure that $\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ is a negation closed W-predicate of type $r \in \{x|A(x,\bar{t})\}$ whenever α_r is in the domain of $\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$.

If $\{x|A(x,\bar{t})\}$ has no E_N predecessors trivially $\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ is a negation closed W-object of type $\{x|A(x,\bar{t})\}$.

To see that $\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ is a W-predicate

$$\vdots$$

note that if $\mathcal{D}$ is on I-form $\dfrac{A(r,\bar{t})}{r \in \{x|A(x,\bar{t})\}}$ and satisfies

$$\vdots$$

$\phi_{\{x|A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$, $A(r,\bar{t})$ will satisfy $\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$ and is $\mathcal{C}$-normalizable by IH, and then so is also $\mathcal{D}$.
So assume we have a deduction

$$\left.\begin{array}{c} \cancel{\neg\, r \in \{x \mid A(x)\}} \\ \vdots \\ \dfrac{\bot}{r \in \{x \mid A(x)\}} \end{array}\right\} \mathcal{D}$$

having the $\bot\left(\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)\right)$ -property.

To see that $\mathcal{D}$ satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ we have to make sure that the contraction

$$\left.\begin{array}{c} \dfrac{\dfrac{\cancel{r \in \{x \mid A(x)\}}}{A(r)} \qquad \dfrac{}{\cancel{\neg A(r)}}}{\dfrac{\bot}{\neg\, r \in \{x \mid A(x)\}}} \\ \vdots \\ \dfrac{\bot}{A(r)} \\ \hline r \in \{x \mid A(x)\} \end{array}\right\} \mathcal{D}_1 \quad \Big\} \mathcal{D}_2$$

satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$.

By IH it is enough to establish that $\mathcal{D}_2$ has the $\bot\left(\phi_{A(x,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})\right)$ -property. This is not hard to establish following the pattern of reasoning used in previous cases.

2.8.2 We also have to prove that for all $\alpha_{v(s)}$ and $\alpha_{v(u)}$ in the domain of $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$,

$$\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)}) = \phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(u)})\left(\mathcal{D}_{s,u}, v(z) \in \{x \mid A(x,\bar{t})\}\right) \quad \text{if} \quad \alpha_{v(s)} = \alpha_{v(u)}\left(\mathcal{D}_{s,u}, v(z)\right).$$

So assume we have a deduction $\mathcal{D}(\Gamma;\ v(s) \in \{x \mid A(x,\bar{t})\})$ which satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(u)})$. Now we have to show that $\dfrac{\mathcal{D} \quad \mathcal{D}_{s,u}}{v(u) \in \{x \mid A(x,\bar{t})\}}$ satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)})$. We have to make sure that the contraction

$$\left.\dfrac{\dfrac{\mathcal{D}}{A(v(s),\bar{t})} \quad \mathcal{D}_{s,u}}{\dfrac{A(v(u),\bar{t})}{v(u) \in \{x \mid A(x,\bar{t})\}}}\right\} \mathcal{D}_1$$

satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(u)})$. Now this is so if $\mathcal{D}_1$ satisfies $\phi_{A(x,\bar{x})}(\alpha_{v(u)}, \bar{\alpha}_{\bar{t}})$. By IH

$$\phi_{A(x,\bar{x})}(\alpha_{v(u)}, \bar{\alpha}_{\bar{t}}) = \phi_{A(x,\bar{x})}(\alpha_{v(s)}, \bar{\alpha}_{\bar{t}})\left(\mathcal{D}_{s,u}, A(v(z),\bar{t})\right).$$

So it will be enough to check that $\dfrac{\mathcal{D}}{A(v(s),\bar{t})}$ satisfies $\phi_{A(x,\bar{x})}(\alpha_{v(s)}, \bar{\alpha}_{\bar{t}})$. $\mathcal{D}$ satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)})$ and thus has the $\mathcal{R}ed\left(\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)})\right)$ -property.

If $\mathcal{D}$ does not reduce to I-form we are done, so assume $\mathcal{D}$ reduces to $\mathcal{D}'$ on I-form satisfying $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)})$:

$$\mathcal{D}''\left\{\begin{array}{c}\vdots\\ A(v(s),\bar{t})\end{array}\right\}\mathcal{D}'$$
$$\frac{}{v(s)\in\{x \mid A(x,\bar{t})\}}$$
$$A(v(s),\bar{t})$$

Now by the definition of $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)})$ we know that $\mathcal{D}''$ satisfies $\phi_{A(x,\bar{x})}(\alpha_{v(s)},\bar{\alpha}_{\bar{t}})$ which means the contraction at the $\mathcal{C}$-main cut satisfies this predicate and so we are done. With obvious changes we use the same argument in the other direction.

2.8.3 Assume $\alpha_{t_i} = \alpha'_{\bar{t}_i}, (\mathcal{D}_{s,u}, t_i(z))$, $i \leq n$.

We must then show that

$$\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}}) = \phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}'_{\bar{t}'})\left(\mathcal{D}_{s,u}, \{x \mid A(x,\bar{t}(z))\}\right).$$

Let $\alpha_{v(s)}$, $\alpha_{v(u)}$ be negation closed W-objects in the domain of $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ and $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ respectively such that $\alpha_{v(s)} = \alpha_{v(u)}(\mathcal{D}_{s,u}, v(z))$.

Now

$$\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_{v(s)}) = \phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}'_{\bar{t}'})(\alpha_{v(u)})\left(\mathcal{D}_{s,u}, , v(z)\in\{x \mid A(x,\bar{t}(z))\}\right)$$

follows by using IH in an argument similar to the one given in 2.8.2. This concludes the definition of $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

3 The substitution property

Given a term or formula $T(x,\bar{x})$, a term $t(\bar{x})$, terms $\bar{t}$ and negation closed W-objects $\bar{\alpha}_{\bar{t}}$ of type $\bar{t}$, then

$$\phi_{T(x,\bar{x})}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right) = \phi_{T(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}}).$$

This can easily be seen by induction on the complexity of $T(x,\bar{x})$:

3.1 $T(x,\bar{x}) \equiv x$.

$$\phi_{T(x,\bar{x})}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right) =_{def.} \phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}) \equiv \phi_{T(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}}).$$

3.2 $T(x,\bar{x}) \equiv x_i$.

$$\phi_{T(x,\bar{x})}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right) =_{def.} \alpha_{t_i} =_{def.} \phi_{T(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}}).$$

3.3 If $T(x,\bar{x})$ is composite, $\phi_{T(x,\bar{x})}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right)$ and $\phi_{T(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ are built up in the same way from W-objects and W-predicates equal by IH. In seeing this note in the $\{x|A(x)\}$ and $\forall$-cases that if z does not actually occur free in t, then $\phi_{t(\bar{x},z)}(\bar{\alpha}_{\bar{t}}, \alpha_u) = \phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}})$ for all negation closed W-objects α_u of type u.

Let us look on some composite examples:

3.3.1 $T(x,\bar{x}) \equiv r(x,\bar{x}) \in u(x,\bar{x})$.

If not $\vdash_N r(t(\bar{t}),\bar{t}) \in u(t(\bar{t}),\bar{t})$, then if $\mathcal{D}$ satisfies $\phi_{r(x,\bar{x})\in u(x,\bar{x})}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right)$, $\mathcal{D}$ is $\mathcal{C}$-normalizable which means $\mathcal{D}$ satisfies $\phi_{r(t(\bar{x}),\bar{x})\in u(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

The same goes for the other direction.
So assume $\vdash_N r(t(\bar{t}),\bar{t}) \in u(t(\bar{t}),\bar{t})$. By IH

$$\phi_{r(x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}}) = \phi_{r(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}}) \quad \text{and}$$

$$\phi_{u(x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}}) = \phi_{u(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}}).$$

So

$$\phi_{u(x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}})(\phi_{r(x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}})) =$$

$$= \phi_{u(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})(\phi_{r(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}}))$$

which gives us what we want.

3.3.2 $T(x,\bar{x}) \equiv \forall_y A(y,x,\bar{x})$.

Assume $\mathcal{D}$ satisfies $\phi_{\forall_y A(y,x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}})$

and so has the $\mathcal{R}ed(\phi_{\forall_y A(y,x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}}))$-

property. If $\mathcal{D}$ is just an assumption or on E-form and does not reduce to I-form we are immediately done. So assume $\mathcal{D}$ is on I-form or reduces to a deduction $\mathcal{D}'$ on I-form:

$$\left.\begin{array}{c} \vdots \\ A(y,t(\bar{t}),\bar{t}) \end{array}\right\} \mathcal{D}_1 \qquad \left.\begin{array}{c} \\ \\ \forall_y A(y,t(\bar{t}),\bar{t}) \end{array}\right\} \mathcal{D}'$$

Now $\mathcal{D}_1(^r/_y)$ satisfies $\phi_{A(y,x,\bar{x})}(\alpha_r,\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}})$

for any r and any negation closed W-object α_r of type r.

As observed above $\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{t(y,\bar{x})}(\alpha_r,\bar{\alpha}_{\bar{t}})$, since y

is not among $\bar{x}$. So $\mathcal{D}_1(^r/_y)$ satisfies $\phi_{A(y,x,\bar{x})}(\alpha_r,\phi_{t(y,\bar{x})}(\alpha_r,\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}})$.

By IH $\mathcal{D}_1\left(\frac{r}{y}\right)$ satisfies $\phi_{A(y,t(y,\bar{x}),\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$

so $\mathcal{D}$ satisfies $\phi_{\forall y A(y,t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

Similar reasoning in the other direction.

3.3.3 $T(x,\bar{x}) \equiv \{z \mid A(z,x,\bar{x})\}$.

We have to show that

$$\phi_{\{z \mid A(z,x,\bar{x})\}}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right)(\alpha_r) = \phi_{\{z \mid A(z,t(\bar{x}),\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$$

for all α_r such that $E_N(r, \{z \mid A(z,t(\bar{t}),\bar{t})\})$.

If $\{z \mid A(z,t(\bar{t}),\bar{t})\}$ has no E_N predecessors

$$\phi_{\{z \mid A(z,x,\bar{x})\}}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right) \underset{\text{def.}}{=} \{z \mid A(z,t(\bar{t}),\bar{t})\} \underset{\text{def.}}{=}$$

$$\underset{\text{def.}}{=} \phi_{\{z \mid A(z,t(\bar{x}),\bar{x})\}}(\bar{\alpha}_{\bar{t}}).$$

So assume $\{z \mid A(z,t(\bar{t}),\bar{t})\}$ has some E_N predecessors and that $\mathcal{D}$ is a deduction that satisfies $\phi_{\{z \mid A(z,x,\bar{x})\}}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right)(\alpha_r)$.

It follows then that $\mathcal{D}$ has the

$$Red\left(\phi_{\{z \mid A(z,x,\bar{x})\}}\left(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}\right)(\alpha_r)\right) \text{- property.}$$

As in the previous case we only have to pay special attention to the situation when $\mathcal{D}$ is on I-form or reduces to a deduction $\mathcal{D}'$ on I-form:

$$\left.\begin{array}{c} \left.\begin{array}{c} \vdots \\ A(r,t(\bar{t}),\bar{t}) \end{array}\right\} \mathcal{D}_1 \\ r \in \{z \mid A(z,t(\bar{t}),\bar{t})\} \end{array}\right\} \mathcal{D}'$$

$\mathcal{D}_1$ satisfies $\phi_{A(z,x,\bar{x})}(\alpha_r, \phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$, z is not among $\bar{x}$, so $\mathcal{D}_1$ will then satisfy

$\phi_{A(z,x,\bar{x})}(\alpha_r, \phi_{t(z,\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$.

By IH $\mathcal{D}_1$ satisfies $\phi_{A(z,t(z,\bar{x}),\bar{x})}(\alpha_r, \bar{\alpha}_{\bar{t}})$ and so $\mathcal{D}'$ satisfies $\phi_{\{z \mid A(t(\bar{x}),\bar{x}\}}(\bar{\alpha}_{\bar{t}})(\alpha_r)$ which gives the desired result.

This concludes the proof of the substitution property.

3.4 We illustrate the use of the substitution property in proving the following corollary to sections 2 and 3 of this chapter:

Let $s(\bar{x})$, $u(\bar{x})$ and $\bar{t}$ be terms such that not $\vdash_N s(\bar{t}) = u(\bar{t})$.

Let $\mathcal{D}(\Gamma; s(\bar{t}) = u(\bar{t}))$ be a deduction on I-form which satisfies $\phi_{s(\bar{x})=u(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

Then for all deductions $\mathcal{D}_1(\Gamma_1; A(s(\bar{t}), \bar{t}))$ and $\mathcal{D}_2(\Gamma_2; A(u(\bar{t}), \bar{t}))$ given a formula $A(x,\bar{x})$, if $\mathcal{D}_1$ satisfies $\phi_{A(s(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ then $\dfrac{\mathcal{D}_1 \quad \mathcal{D}}{A(u(\bar{t}),\bar{t})}$ satisfies $\phi_{A(u(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ and if $\mathcal{D}_2$ satisfies $\phi_{A(u(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$, then $\dfrac{\mathcal{D}_2 \quad \mathcal{D}}{A(s(\bar{t}),\bar{t})}$ satisfies $\phi_{A(s(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

This is easily proved by induction on the construction of $A(x,\bar{x})$ treating the $\in$ -cases as basis cases.

We just follow suitable parts of the arguments given in 2.3 - 2.7 establishing $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{T(\bar{x})}(\alpha'_{\bar{t}})\,(\mathcal{D}_{S,u}, T(\bar{t}(x)))$.

In the $\in$ -cases we note that in all cases except when $A(x,\bar{x}) \equiv t(\bar{x}) \in X$ there will be no contraction defined at the $S(\bar{t}) = u(\bar{t})$ -cut, since not $\vdash_N S(\bar{t}) = u(\bar{t})$.

Assume $A(x,\bar{x}) \equiv t(\bar{x}) \in X$. By assumption $\mathcal{D}_1$ satisfies $\phi_{t(\bar{x}) \in S(\bar{x})}(\bar{\alpha}_{\bar{t}})$. Using the substitution property we see that $\mathcal{D}_1$ satisfies $\phi_{x \in S(\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$.

Now $\mathcal{D}$ satisfies $\phi_{S(\bar{x}) = u(\bar{x})}(\bar{\alpha}_{\bar{t}})$, so the contraction at the $\mathcal{C}$ -main cut $S(\bar{t}) = u(\bar{t})$ in $\dfrac{\mathcal{D}_1 \quad \mathcal{D}}{t(\bar{t}) \in u(\bar{t})}$ will satisfy $\phi_{x \in u(\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$. Using the substitution property once more we see that the contraction satisfies $\phi_{t(\bar{x}) \in u(\bar{x})}(\bar{\alpha}_{\bar{t}})$ as well.

To establish the other direction we use the same arguments with obvious changes.

4 Every deduction in N satisfies some negation closed W-predicate

Given a deduction

$$\left.\begin{array}{c} A_1(\bar{x}) \quad \ldots \quad A_n(\bar{x}) \\ \vdots \\ A(\bar{x}) \end{array}\right\} \mathcal{D}$$

By induction on the length of $\mathcal{D}$ I will show for any $\bar{t}$ and any negation closed W-objects $\bar{\alpha}_{\bar{t}}$ of types $\bar{t}$ that if

$$\vdots$$
$$A_i(\bar{t}) \text{ , } i \leq n,$$

satisfies $\phi_{A_i(\bar{x})}(\bar{\alpha}_{\bar{t}})$, then

$$\left.\begin{array}{c} \vdots \quad\quad \vdots \\ A_1(\bar{t}) \ldots A_n(\bar{t}) \\ \vdots \\ A(\bar{t}) \end{array}\right\} \mathcal{D}^*$$

satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.1 If $\mathcal{D}$ is $A(\bar{x})$, then the claim is obviously true.

4.2 If $\mathcal{D}$ is on I-form we have to check a number of cases:

4.2.1 $\mathcal{D}$ is on $\in$ I-form

$$\left.\begin{array}{c} \left.\begin{array}{c} \vdots \\ A(r(\bar{x}),\bar{x}) \end{array}\right\} \mathcal{D}_1 \\ \hline r(\bar{x}) \in \{x \mid A(x,\bar{x})\} \end{array}\right\} \mathcal{D}$$

By IH $\mathcal{D}_1^*$ satisfies $\phi_{A(r(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ and so by the substitution

property also $\phi_{A(x,\bar{x})}(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}),\bar{\alpha}_{\bar{t}})$ and so $\mathcal{D}_1^*$ is $\mathcal{C}$-normalizable which if not $\vdash_N r(\bar{t}) \in \{x \mid A(x,\bar{t})\}$ directly gives us that $\mathcal{D}^*$ satisfies $\phi_{r(\bar{x}) \in \{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$.

If $\vdash_N r(\bar{t}) \in \{x \mid A(x,\bar{t})\}$, then we know by IH that $\mathcal{D}^*$ satisfies $\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ and so satisfies $\phi_{r(\bar{x}) \in \{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$.

4.2.2 $\mathcal{D}$ is on $=$ I-form

$$\mathcal{D}_1 \left\{ \begin{array}{cc} \cancel{x \in r(\bar{x})} & \cancel{x \in t(\bar{x})} \\ \vdots & \vdots \\ x \in t(\bar{x}) & x \in r(\bar{x}) \end{array} \right\} \mathcal{D}_2 \Bigg\} \mathcal{D}$$
$$r(\bar{x}) = t(\bar{x})$$

By IH for any term v and any negation closed W-object α_v of type v

$$\left.\begin{array}{c} \vdots \\ v \in r(\bar{t}) \\ \vdots \\ v \in t(\bar{t}) \end{array}\right\} \mathcal{D}_1^*(v/x) \qquad \left.\begin{array}{c} \vdots \\ v \in t(\bar{t}) \\ \vdots \\ v \in r(\bar{t}) \end{array}\right\} \mathcal{D}_2^*(v/x)$$

satisfy $\phi_{x \in t(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$ and $\phi_{x \in r(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$ respectively

if $\begin{array}{c} \vdots \\ v \in r(\bar{t}) \end{array}$, $\begin{array}{c} \vdots \\ v \in t(\bar{t}) \end{array}$ satisfy $\phi_{x \in r(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$ and

$\phi_{x \in t(\bar{x})}(\alpha_v, \bar{\alpha}_{\bar{t}})$ respectively. So $\mathcal{D}^*$ satisfies $\phi_{r(\bar{x}) = t(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.2.3 $\mathcal{D}$ is on & I-form

$$\left.\dfrac{\begin{array}{c}\mathcal{D}_1\left\{\begin{array}{cc}\vdots & \vdots\\ A(\bar{x}) & B(\bar{x})\end{array}\right\}\mathcal{D}_2\\ A(\bar{x})\&B(\bar{x})\end{array}}{}\right\}\mathcal{D}$$

By IH $\mathcal{D}_1^*$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\mathcal{D}_2^*$ satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$

so $\mathcal{D}^*$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.2.4 $\mathcal{D}$ is on $\rightarrow$ I-form

$$\left.\dfrac{\begin{array}{c}\not{A(\bar{x})}\\ \vdots\\ B(\bar{x})\end{array}\Big\}\mathcal{D}_1}{A(\bar{x})\rightarrow B(\bar{x})}\right\}\mathcal{D}$$

By IH for any $\mathcal{D}_2$ satisfying $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$

$$\left.\begin{array}{c}\vdots\\ A(\bar{t})\end{array}\right\}\mathcal{D}_2$$

$$\left.\begin{array}{c}\vdots\\ B(\bar{t})\end{array}\right\}\mathcal{D}_1^*$$

satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$, which means $\mathcal{D}^*$ satisfies $\phi_{A(\bar{x})\rightarrow B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.2.5 $\mathcal{D}$ is on $\forall$ I-form

$$\left.\dfrac{\begin{array}{c}\vdots\\ A(x,\bar{x})\end{array}\Big\}\mathcal{D}_1}{\forall x\, A(x,\bar{x})}\right\}\mathcal{D}$$

By IH for any t and any negation closed W-object α_t of type t

$$\left.\begin{array}{c}\vdots\\ A(t,\bar{t})\end{array}\right\} \mathcal{D}_1^*(t/x) \quad \text{satisfies} \quad \phi_{A(x,\bar{x})}(\alpha_t, \bar{\alpha}_{\bar{t}})$$

so $\mathcal{D}^*$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.3 $\mathcal{D}$ is on E-form.

As in 4.2 we have to go through the different possible cases.

4.3.1 $\mathcal{D}$ is on $\in$ E-form

$$\left.\begin{array}{c}\left.\begin{array}{c}\vdots\\ r(\bar{x}) \in \{x \mid A(x,\bar{x})\}\end{array}\right\} \mathcal{D}_1 \\ \hline A(r(\bar{x}),\bar{x})\end{array}\right\} \mathcal{D}$$

By IH $\mathcal{D}_1^*$ satisfies $\phi_{r(\bar{x}) \in \{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ and so has the

$Red(\phi_{r(\bar{x}) \in \{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}}))$ - property.

In all cases except when $\mathcal{D}_1^*$ reduces to I-form we are immediately done so assume $\mathcal{D}_1^*$ reduces to I-form

$$\left.\begin{array}{c}\left.\begin{array}{c}\vdots\\ A(r(\bar{t}),\bar{t})\end{array}\right\} \mathcal{D}_2 \\ \hline r(\bar{t}) \in \{x \mid A(x,\bar{t})\}\end{array}\right\} \mathcal{D}_1^{*'}$$

Now if not $\vdash_N r(\bar{t}) \in \{x \mid A(x,\bar{t})\}$, $\mathcal{D}_1^{*'}$ will be the $\mathcal{C}$-main deduction in $\dfrac{\mathcal{D}_1^{*'}}{A(r(\bar{t}),\bar{t})}$

By IH $\mathcal{D}_1^{*'}$ satisfies $\phi_{r(\bar{x}) \in \{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$ and so is $\mathcal{C}$-normalizable.

So in this case we are done.

If $\vdash_N r(\bar{t}) \in \{x \mid A(x,\bar{t})\}$ we know that $\mathcal{D}_1^{*'}$ satisfies

$\phi_{\{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}))$, since by IH $\mathcal{D}_1^{*'}$ satisfies

$\phi_{r(\bar{x}) \in \{x \mid A(x,\bar{x})\}}(\bar{\alpha}_{\bar{t}})$. So $\mathcal{D}_2$ satisfies $\phi_{A(x,\bar{x})}(\phi_{r(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$.

Using the substitution property we see that $\mathcal{D}_2$ satisfies

$\phi_{A(r(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$, so this is also the W-predicate the

contraction at the $r(\bar{t}) \in \{x \mid A(x,\bar{t})\}$ -cut will satisfy. This

means that $\mathcal{D}^*$ satisfies $\phi_{A(r(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$, which is what

we want.

4.3.2 $\mathcal{D}$ is on $=$E-form

$$\left.\begin{array}{c} \mathcal{D}_1 \left\{\begin{array}{c} \vdots \qquad\qquad \vdots \\ A(s(\bar{x}),\bar{x}) \qquad s(\bar{x}) = u(\bar{x}) \end{array}\right\} \mathcal{D}_2 \\ \hline A(u(\bar{x}),\bar{x}) \end{array}\right\} \mathcal{D} \qquad (A(x))$$

So we have to make sure that

$$\left.\begin{array}{c} \mathcal{D}_1^* \left\{\begin{array}{c} \vdots \qquad\qquad \vdots \\ A(s(\bar{t}),\bar{t}) \qquad s(\bar{t}) = u(\bar{t}) \end{array}\right\} \mathcal{D}_2^* \\ \hline A(u(\bar{t}),\bar{t}) \end{array}\right\} \mathcal{D}^*$$

satisfies $\phi_{A(u(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

By IH $\mathcal{D}_1^*$ satisfies $\phi_{A(s(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\mathcal{D}_2^*$ satisfies

$\phi_{s(\bar{x})=u(\bar{x})}(\bar{\alpha}_{\bar{t}})$. So $\mathcal{D}_2^*$ has the $Red(\phi_{s(\bar{x})=u(\bar{x})}(\bar{\alpha}_{\bar{t}}))$-

property.

If $\mathcal{D}_2^*$ reduces to a deduction on E-form having no $\mathcal{C}$-main cut or to just an assumption we are done. So assume $\mathcal{D}_2^*$ reduces to a deduction $\mathcal{D}_{s(\bar{t}),u(\bar{t})}$ on I-form satisfying $\phi_{s(\bar{x})=u(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

If not $\vdash_N s(\bar{t})=u(\bar{t})$, then we use 3.4 of this chapter to see that

$$\left.\begin{array}{c} \mathcal{D}_1^* \quad \mathcal{D}_{s(\bar{t}),u(\bar{t})} \\ \hline A(u(\bar{t}),\bar{t}) \end{array}\right\} \mathcal{D}_3$$

satisfies $\phi_{A(u(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

So assume $\vdash_N s(\bar{t})=u(\bar{t})$. Now since $\mathcal{D}_{s(\bar{t}),u(\bar{t})}$ satisfies $\phi_{s(\bar{x})=u(\bar{x})}(\bar{\alpha}_{\bar{t}})$ it is not difficult to see that $\mathcal{D}_{s(\bar{t}),u(\bar{t})}$ satisfies the conditions given in 1.3 of this chapter and that $\phi_{s(\bar{x})}(\bar{\alpha}_{\bar{t}}) = \phi_{u(\bar{x})}(\bar{\alpha}_{\bar{t}})(\mathcal{D}_{s(\bar{t}),u(\bar{t})}, x)$. So by section 2 of this chapter we see that

$$\phi_{A(x,\bar{x})}(\phi_{s(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}}) = \phi_{A(x,\bar{x})}(\phi_{u(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})(\mathcal{D}_{s(\bar{t}),u(\bar{t})}, A(x,\bar{t})).$$

By IH $\mathcal{D}_1^*$ satisfies $\phi_{A(s(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$, so using the substitution property we see that $\mathcal{D}_1^*$ satisfies $\phi_{A(x,\bar{x})}(\phi_{s(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$.

So $\mathcal{D}_3$ satisfies $\phi_{A(x,\bar{x})}(\phi_{u(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$ and using the substitution property once more we see $\mathcal{D}_3$ satisfies $\phi_{A(u(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ which is what we want.

4.3.3 $\mathcal{D}$ is on & E-form

$$\left.\begin{array}{c}\left.\begin{array}{c}\vdots\\ A(\bar{x})\&B(\bar{x})\end{array}\right\}\mathcal{D}_1\\ \hline A(\bar{x})\end{array}\right\}\mathcal{D}$$

By IH $\mathcal{D}_1^*$ satisfies $\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and so has the

$Red(\phi_{A(\bar{x})\&B(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ -property. As in previous cases we are mainly interested in the situation when $\mathcal{D}_1^*$ reduces to I-form

$$\mathcal{D}_2\left\{\begin{array}{c}\begin{array}{cc}\vdots & \vdots\\ A(\bar{t}) & B(\bar{t})\end{array}\\ \hline A(\bar{t})\&B(\bar{t})\end{array}\right\}\mathcal{D}_1^{*'}$$

Now we know that $\mathcal{D}_2$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and so does then the contraction of $\dfrac{\mathcal{D}_1^{*'}}{A(\bar{t})}$ which is what we want to see that $\mathcal{D}^*$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.3.4 $\mathcal{D}$ is on $\rightarrow$ E-form

$$\mathcal{D}_1\left\{\begin{array}{c}\begin{array}{cc}\vdots & \vdots\\ A(\bar{x}) & A(\bar{x})\rightarrow B(\bar{x})\end{array}\\ \hline B(\bar{x})\end{array}\right\}\mathcal{D}_2\Bigg\}\mathcal{D}$$

Now by IH $\mathcal{D}_1^*$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\mathcal{D}_2^*$ satisfies $\phi_{A(\bar{x})\rightarrow B(\bar{x})}(\bar{\alpha}_{\bar{t}})$. To see that $\mathcal{D}^*$ satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$, as in

previous cases, we are mainly interested in the case when $\mathcal{D}_2^*$ reduces to I-form so that we have a deduction

$$\dfrac{\mathcal{D}_1^* \qquad \left.\begin{array}{c} \not{A(\bar{t})} \\ \vdots \\ \dfrac{B(\bar{t})}{A(\bar{t}) \to B(\bar{t})} \end{array}\right\} \mathcal{D}_2^{*\prime}}{B(\bar{t})}$$

That the contraction at the $\mathcal{C}$-main cut satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ follows from the fact that $\mathcal{D}_2^{*\prime}$ satisfies $\phi_{A(\bar{x}) \to B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and $\mathcal{D}_1^*$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$, so $\mathcal{D}^*$ satisfies $\phi_{B(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.3.5 $\mathcal{D}$ is on $\forall$ E-form

$$\left.\begin{array}{c} \vdots \\ \dfrac{\forall x A(x,\bar{x})}{A(t(\bar{x}),\bar{x})} \end{array}\right\} \mathcal{D}_1 \Bigg\} \mathcal{D}$$

By IH $\mathcal{D}_1^*$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$.

Now as in previous cases assume we have a reduction of the major deduction to a deduction on I-form. In this case then

$$\left.\begin{array}{c} \vdots \\ \dfrac{A(x,\bar{t})}{\forall x A(x,\bar{t})} \end{array}\right\} \mathcal{D}_2 \Bigg\} \mathcal{D}_1^{*\prime}$$

$\mathcal{D}_1^{*'}$ satisfies $\phi_{\forall x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ so for any t and any negation closed W-object α_t of type t, $\mathcal{D}_2(t/x)$ satisfies $\phi_{A(x,\bar{x})}(\alpha_t, \bar{\alpha}_{\bar{t}})$.

So in particular $\mathcal{D}_2(t(\bar{t})/x)$ satisfies $\phi_{A(x,\bar{x})}(\phi_{t(\bar{x})}(\bar{\alpha}_{\bar{t}}), \bar{\alpha}_{\bar{t}})$ and then by the substitution property $\mathcal{D}_2(t(\bar{t})/x)$ satisfies $\phi_{A(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$ which is what we want to see that the contraction of $\dfrac{\mathcal{D}_1^{*'}}{A(t(\bar{t}),\bar{t})}$ satisfies $\phi_{A(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

So $\mathcal{D}^*$ satisfies $\phi_{A(t(\bar{x}),\bar{x})}(\bar{\alpha}_{\bar{t}})$.

4.3.6 $\mathcal{D}$ is on $\perp$ E-form

$$\dfrac{\left.\begin{array}{c}\neg A(\bar{x}) \\ \vdots \\ \perp\end{array}\right\}\mathcal{D}_1}{A(\bar{x})}\Bigg\}\mathcal{D}$$

By section 2 of this chapter to see that $\mathcal{D}^*$ satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ it is enough to convince ourselves that $\mathcal{D}^*$ has the $\perp(\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}}))$-property.

Firstly by IH $\mathcal{D}_1^*$ is $\mathcal{C}$-normalizable, since by IH $\mathcal{D}_1^*$ satisfies $\phi_{\perp}(\bar{\alpha}_{\bar{t}})$.

So assume the deduction

$$\left.\begin{array}{c}A(\bar{t}) \\ \vdots \\ \perp\end{array}\right\}\mathcal{D}_2$$
$$\neg A(\bar{t})$$

is such that

$$\begin{array}{c} \vdots \\ A(\bar{t}) \\ \vdots \\ \bot \end{array}$$

is $\mathcal{C}$-normalizable for all $\begin{array}{c} \vdots \\ A(\bar{t}) \end{array}$ satisfying $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$.

This means simply that $\mathcal{D}_a$ satisfies $\phi_{\neg A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ and by IH clearly

$$\dfrac{\begin{array}{c} \cancel{A(\bar{t})} \\ \vdots \\ \bot \end{array}}{\neg A(\bar{t})}$$

$$\vdots$$

$$\bot$$

satisfies $\phi_{\bot}(\bar{\alpha}_{\bar{t}})$, and so is $\mathcal{C}$-normalizable. So $\mathcal{D}^*$ has the

$\bot\left(\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})\right)$ -property and consequently satisfies

$\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$ which is what we want.

To conclude we then have the following:

Theorem: If N is a wellfounded fragment of naive set theory, then every deduction in N is $\mathcal{C}$-normalizable.

5 $\mathfrak{C}$-normalizability for wellfounded fragments of naive set theory with intuitionistic logic.

If we leave out in sections 1 - 4 of this chapter everything that has to do with classical $\perp E$ inferences, that is leave out the question whether W-objects and W-predicates are negation closed or not, then we will have a treatment of wellfounded fragments of naive set theory with intuitionistic logic based on $\perp, \in, =, \&, \rightarrow, \forall$. Now $\vee$ and $\exists$ will be definable in all such fragments of naive set theory in which we have at least a weak form of Zermelo's subset axiom available. If we check the proofs given in I, 1.3.3 we see that what we need is for formulas C at least the term $\{x \mid x \in r \,\&\, C\}$ for some term r. So in this sense here we have already covered all ordinary fragments. So much for the pragmatic aspect. If we want a more general treatment it is not hard to see how we can extend the arguments given in 1 - 4 of this chapter to cover $\vee$ and $\exists$ where the definition of $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}})$ will be extended in the following way:

5.1 $\phi_{T(\bar{x})}(\bar{\alpha}_{\bar{t}})$

5.1.1 $T(\bar{x}) \equiv A(\bar{x}) \vee B(\bar{x})$

$A(\bar{t}) \vee B(\bar{t})$ satisfies $\phi_{A(\bar{x}) \vee B(\bar{x})}(\bar{\alpha}_{\bar{t}})$

If $\mathcal{D}(\Gamma;\ A(\bar{t}) \vee B(\bar{t}))$ is on I-form

$$\vdots$$

$$\frac{A(\bar{t})}{A(\bar{t}) \vee B(\bar{t})}$$

$\mathcal{D}$ satisfies $\phi_{A(\bar{x}) \vee B(\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff $\vdots$ $A(\bar{t})$

satisfies $\phi_{A(\bar{x})}(\bar{\alpha}_{\bar{t}})$, and similarly if $\mathcal{D}$ is

$$\frac{\begin{array}{c}\vdots\\ \mathcal{B}(\bar{t})\end{array}}{A(\bar{t})\vee\mathcal{B}(\bar{t})}$$

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{A(\bar{x})\vee\mathcal{B}(\bar{x})}(\bar{\alpha}_{\bar{t}})$

iff $\mathcal{D}$ has the $E(\phi_{A(\bar{x})\vee\mathcal{B}(\bar{x})}(\bar{\alpha}_{\bar{t}}))$ - property.

5.1.2 $T(\bar{x})\equiv\exists x A(x,\bar{x})$

$\exists x A(x,\bar{t})$ satisfies $\phi_{\exists x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$

If $\mathcal{D}(\Gamma;\ \exists x A(x,\bar{t}))$ is on I-form

$$\frac{\begin{array}{c}\vdots\\ A(r,\bar{t})\end{array}\Big\}\mathcal{D}_1}{\exists x A(x,\bar{t})}$$

then $\mathcal{D}$ satisfies $\phi_{\exists x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ if there is a W-object α_r of type r such that $\mathcal{D}_1$ satisfies $\phi_{A(x,\bar{x})}(\alpha_r,\bar{\alpha}_{\bar{t}})$.

If $\mathcal{D}$ is on E-form, then $\mathcal{D}$ satisfies $\phi_{\exists x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}})$ iff

$\mathcal{D}$ has the $E(\phi_{\exists x A(x,\bar{x})}(\bar{\alpha}_{\bar{t}}))$ - property.

So wellfounded fragments of naive set theory with intuitistic logic will also have the property that all deductions $\mathcal{D}$ are $\mathcal{C}$ -normalizable.

5.2 Let N be a wellfounded fragment of naive set theory with intuitionistic logic. By 5.1 of this chapter and 1.4 of chapter II it follows that:

5.2.1 If there is a proof in N of $A \vee B$ then there is a proof of A in N or a proof of B in N.

5.2.2 If there is a proof of $\exists x A(x)$ in N, then there is a proof of $A(t)$ in N for some closed term t.

5.2.1 and 5.2.2 follow from the fact that closed $\mathcal{C}$-normal deductions in N are on I-form. We know that the proofs referred to in 5.2.1 and 5.2.2 can be obtained by $\mathcal{C}$-normalizing the given proofs of $A \vee B$ and $\exists x A(x)$. Unfortunately, in this context $\mathcal{C}$ is not a decidable predicate for ordinary N. But if we instead choose a predicate $\mathcal{C}_1$ defined by

$$\mathcal{C}_1(\mathcal{D}) \quad \text{iff } \mathcal{D} \text{ is closed}$$

we know by 6.5.6 of chapter I that all deductions in N are $\mathcal{C}_1$-normalizable. By 1.3.3 of chapter II closed $\mathcal{C}_1$-normal deductions are on I-form so we obtain the results 5.2.1 and 5.2.2 by $\mathcal{C}_1$-normalizing the given proofs of $A \vee B$ and $\exists x A(x)$ where $\mathcal{C}_1$ is now a decidable predicate.

6 Notes

The proof given in this chapter that deductions in a wellfounded fragment of naive set theory are $\mathcal{C}$-normalizable is essentially an extension to set theory with extensionality and classical logic of a proof of normalization in simple type theory without extensionality and with intuitionistic logic due independently to Girard ([Gi]) and Martin-Löf ([M3]). The treatment of the classical $\perp E$-rule given here follows Prawitz treatment of this rule in second order logic ([Pr5] pp 18-20).

The proofs of Girard, Martin-Löf and Prawitz are based on Martin-Löf's adaptation (See [M1]) of a method for normalizing terms due to Tait ([T]) and Girard's idea to use wellfoundedness objects and wellfoundedness predicates ("Candidat de rêductibilité" in Girard's terminology) to overcome the problems involved in higher order quantification.

If one is only interested in transforming closed deductions $\mathcal{D}$ into closed deductions $\mathcal{D}'$ on I-form one can use a simpler formulation of negation-closedness than the one given in this chapter (1.2):

α_A is closed under negation if for all

$$\left.\begin{array}{c} \neg A \\ \vdots \\ \dfrac{\perp}{A} \end{array}\right\} \mathcal{D}$$

$\mathcal{D}$ satisfies α_A.

One then needs to postulate that deductions ending with atomic formulas are $\mathcal{C}$-normalizable, and use semantical information to prove that there are no closed deductions with atomic conclusions. In this case we have a rather weak analysis of the form of $\mathcal{C}$-normal deductions, but it will still be true that closed $\mathcal{C}$-normal deductions are on I-form.

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